Shock and Vibration Digest, 30: 106–113 (1998)

Reliability of Vibrating Structures With Uncertain Inputs

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Abstract

We evaluate the reliability of vibrating systems subject to severely deficient information about the dynamic loads. We stress non-probabilistic information-gap models of uncertainty, which are adapted to severe lack of information. When some probabilistic information is available, we show how it can be incorporated in a hybrid probabilistic/info-gap analysis. We outline the theory of robust reliability, which replaces probabilistic reliability in those situations where prior information is insufficient to verify the choice of a probability density. We also illustrate a hybrid probabilistic/info-gap reliability analysis. Finally, we use the "gambler's theorem" and the idea of aversion to risk to provide an overall quantitative assessment of the performance of a system in an uncertain environment.

1 Modelling the Unknown

"Prediction", said Niels Bohr, "is always difficult, especially of the future." But we act all the time on suppositions extrapolated from incomplete information. From coin-flips to international conflicts, we predict outcomes based on partial information. When we have extensive experience, like in ambient vibrations under known and controlled conditions, we can make reliable assertions. But in unique and unfamiliar circumstances we have severely limited prior knowledge, so we must be much more circumspect.

In analyzing the reliability of critical components and systems with respect to rare and extraordinary events, about which we know very little, we must avoid unverifiable assumptions as much as possible. In particular, we must represent the uncertainties as reliably as possible, without extraneous assumptions. In this paper we discuss a method of reliability analysis which is developed for this purpose. There is no free lunch, informationally speaking, so an analysis based on limited prior information will be able to make only modest predictions. However, the crucial point is that the analysis itself be reliable and not illusory.

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Analysis and prediction under uncertainty depend upon representation of what is known about the uncertain phenomenon. The choice of a model of uncertainty depends on the type and quantity of information which is available. Three classes of uncertainty-models are current today. **Probability models** describe frequency of recurrence of events, or, equivalently, subjective degrees of belief of these likelihoods. **Information-gap models** of uncertainty quantify the disparity between what is known, and what needs to be known in order to make an optimal decision. **Fuzzy models** portray linguistic ambiguities as well as assessments of possibility or necessity of occurrence.

Most people have flipped enough coins to have considerable confidence that the chance of 'heads' on the next flip is exceedingly close to 1/2. Quite reasonably, we extrapolate from the vast ensemble of past experience. But if I tell you that I have in my pocket a coin confiscated by the police from an illicit gambling joint, you may balk before betting that this is a 'fair' coin with equal probabilities for 'heads' and 'tails'. For this coin, due to the paucity of information, you are unable to *rationally* choose a probability model to describe the frequency of outcomes.

The *rational* thing to do is to keep away from underground gambling establishments. However, in many technological situations the uncertainties are so complex, and the resources of time and money so limited, that we face severe lack of information. In analyzing the dynamics of critical components in innovative technology intended for new applications in unusual and uncontrolled environments, we may be unable to test or verify the choice of a probabilistic model. For example, the vibrations induced in ordinary highway driving can, with diligence, be measured and modelled accurately with probabilistic models. Off-road vehicles, however, may be subjected to extreme loads which are quite unpredictable based either on laboratory tests or ordinary automotive experience. Unless system-specific data on the relevant terrain is available, it is unrealistic (and probably unsafe) to extrapolate from the experience of ordinary driving. In situations such as this we use an information-gap model to quantify the disparity between the firm data in hand, and the missing information which is needed for a reliable decision. In the next section we will show how info-gap models are constructed and used in vibration analysis.

The transition from a probabilistic to an info-gap model of uncertainty is motivated by a severe lack of data. But some uncertainties are related to the elasticity of language rather than to the limitations of firm knowledge. Imagine an expert who listens to the hum of a failing pump, and says "Ah ha! You hear that rattle followed by the click-click? That means you must tighten the bearings." But what is a 'rattle', what exactly is a 'click-click', and how 'tight' is 'tight'? Here the uncertainty in the information provided by this expert comes from the wonderfully useful but frustrating elasticity of language. Even the expert himself, so long as he remains on the level of verbal advice, will be hard pressed to dispel the uncertainty which pervades his language. While linguistic information is very important in many technical applications, and though its uncertainties are subtle and interesting, we will say no more about them in this paper.

2 Information-Gap Models of Uncertainty

An information-gap model is a family of nested sets. Each set corresponds to a particular degree of uncertainty, according to its level of nesting. Each element in a set represents a possible realization of the uncertain event. Info-gap models, and especially convex-set models of uncertainty, have been described elsewhere, both technically (Ben-Haim, 1985a, 1996; Ben-Haim and Elishakoff, 1990; Elishakoff and Zhu, 1994), non-technically (Ben-Haim, 1994a) and axiomatically (Ben-Haim, 1998).

Consider only one simple technological example: uncertain forces $u(t)$ exerted on a building during an earthquake. (Later we will consider more examples). The temporal variation of the force resulting from a nominal or typical seismic excitation is $\widetilde{u}(t)$ which is a known function. The actual load $u(t)$ deviates by an unknown amount from the nominal load $\widetilde{u}(t)$. The set of all excitation-functions $u(t)$ whose energy of deviation from $\widetilde{u}(t)$ is bounded by α^2, is one set in a family of nested sets:

$$\mathcal{U}(\alpha, \widetilde{u}) = \left\{ u(t) : \int_0^\infty [u(t) - \widetilde{u}(t)]^2 \, \mathrm{d}t \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{1}$$

This info-gap model, $\mathcal{U}(\alpha, \widetilde{u})$, is a family of nested sets for $\alpha \ge 0$. This means that $\mathcal{U}(\alpha, \widetilde{u})$ is a subset of $\mathcal{U}(\beta, \widetilde{u})$ if $\alpha \le \beta$. Uncertainty is expressed at two levels by this info-gap model. For fixed α, the set $\mathcal{U}(\alpha, \widetilde{u})$ represents a degree of uncertain variability of the load function $u(t)$. The greater the value of α, the greater the variation, so α is called the *uncertainty parameter* and expresses the information gap between what is known ($\widetilde{u}(t)$ in the above example) and what needs to be known for an ideal solution (the exact function $u(t)$). The value of α is usually unknown, which constitutes the second level of uncertainty.

Each set in the family of uncertainty-sets defined in eq.(1) is in fact a convex set. The convexity of the sets has not been assumed; the convexity simply arises as a by-product of how the partial information is quantified. Convex info-gap models are commonly encountered, and are called *convex models*.

In the info-gap model of eq.(1), each set in the family is defined as the collection of all functions consistent with the prior information, up to uncertainty α. This is characteristic of how info-gap models are formulated in general. In this way, the info-gap model is constructed with very parsimonious use of information.

3 Vibration Analysis With Uncertain Inputs

Sets of functions were used for representing uncertainty in shock and vibration studies long before the current general formulation of info-gap models emerged. In the late 1960's Drenick (1968, 1970) studied the design of structures subject to uncertain seismic excitation. Even though the specific waveform of the ground motion was unknown, Drenick supposed that the total energy of the ground motion would be bounded, and would be proportional to the integral of the square of the motion. While the ground displacement, $x(t)$, was unknown, it was constrained by the total energy, E:

$$\int_{-\infty}^{\infty} x^2(t) \, \mathrm{d}t \le E \tag{2}$$

Thus the uncertainty in the ground motion was represented as a set of allowed functions $x(t)$ in a manner quite similar to the info-gap model of eq.(1).

The only substantial difference between Drenick's model and eq.(1) is that the latter considers a family of nested sets, $\mathcal{U}(\alpha, \widetilde{u})$ for $\alpha \ge 0$, while Drenick concentrated on a single member of that family of sets: $\alpha^2 = E$. This distinction will become significant when we come to consider the reliability of structures subject to uncertain input.

Shinozuka (1970), working at the same time as Drenick, employed what would now be called a Fourier-envelope info-gap model. Defining $X(\omega)$ as the Fourier transform of the input ground motion $x(t)$, Shinozuka supposed that the modulus of $X(\omega)$ is constrained within a known envelope $X_{\mathrm{env}}(\omega)$:

$$|X(\omega)| \le X_{\mathrm{env}}(\omega) \tag{3}$$

While the actual input spectrum $X(\omega)$ is not known, the uncertain variation is constrained by the known envelope. This defines a set of possible input spectra, and this set is one in the family of uncertainty sets constituting the Fourier-envelope info-gap model:

$$\mathcal{X}(\alpha, \overline{X}) = \left\{ X(\omega) : |X(\omega) - \overline{X}(\omega)| \le \alpha X_{\mathrm{env}}(\omega) \right\}, \quad \alpha \ge 0 \tag{4}$$

Substantial and extensive work using set-models of uncertainty was done around the same time outside the discipline of structural vibrations. Schweppe (1973), Witsenhausen (1968a,b), Chernousko (1980, 1983) and others used set-models in control theory and state estimation.

What characterizes all of these applications is that sets of uncertain vectors or functions are formulated to represent unknown variation of physical quantities. This work testifies to the intuitive attractiveness and practical utility of set-models of uncertainty.

Slightly later (Ben-Haim, 1985a,b; Ben-Haim and Shenhav, 1983, 1984; Shenhav and Ben-Haim, 1984) appeared what was perhaps the first indication that the work of these early pioneers was in fact a collection of examples of a more general theory of uncertainty, which we know today as convex modelling or info-gap models of uncertainty. While the early workers had exploited to great advantage the fact that their uncertainty sets where (almost invariably) convex, the 'convexity theorem' (Ben-Haim, 1985a) established a mathematical connection between set-convexity and uncertainty. That the convexity theorem was developed in yet another field — optimization of material assay — shows how widespread is the need for alternative theories of uncertainty. This result was extended and expressed as a limit theorem by Ben-Haim and Elishakoff (1990).

Since the early 1990's we find convex-set info-gap models being applied to a wide range of problems in mechanics, many of which are summarized by Ben-Haim and Elishakoff (1990). Givoli and Elishakoff (1992) use info-gap models to analyze stress concentrations in plates with irregularly shaped holes. Lindberg (1992a,b) applies info-gap models in the analysis of dynamic pulse buckling of thin-walled shells. Among his other conclusions, Lindberg notes the tremendous saving in time using set-models instead of monte-carlo-based probabilistic models, without loss of fidelity to available prior information. Natke and Soong (1993) examine structural optimization by using info-gap models to represent uncertain dynamic loads. Elishakoff and Zhu (1994) use info-gap models to study acoustically excited structures. Ben-Haim (1994b) uses info-gap models to study the fatigue life of structures with uncertain inputs. Ben-Haim, Chen and Soong (1996) use convex info-gap models in the analysis of seismic safety, comparing their results to probabilistic analysis. Pantelides and Tzan (1996) as well as Tzan and Pantelides (1996a,b) study the safety and active control of buildings with uncertain ground loadings represented by info-gap models. Qiu and Gu (1996) study trusses and plates with uncertain structural and material parameters. Pantelides (1996) studies the stability of an elastic bar on a foundation with uncertain stiffness using convex models of uncertainty. More recently, the range of application of set-theoretic info-gap models of uncertainty has grown to include sequential analysis (Ben-Haim, 1998) and inference from uncertain evidence (Ben-Haim, 1997b), as well as project management under uncertainty (Ben-Haim and Laufer).

In the following section we will discuss the use of info-gap models in a theory of reliability which employs no probabilistic models and is particularly suited to situations plagued by severe lack of information. Before that, however, we examine several examples of vibration analysis with info-gap models of uncertainty.

Example 1 *Extremal displacement with uncertain input.* Let us consider a simple but characteristic vibration problem with uncertain transient input. Given very limited prior information about the inputs, we wish to estimate the extremal displacements of the vibrating system.

A typical situation is that we know which frequencies make up the input function, and we have rough global data about the relative range of variation of the amplitudes of these frequencies. However, we lack sufficient data to express the relative likelihoods of different inputs. With such exiguous information we can not reasonably select a probabilistic model for the dynamic loads. We can, however, employ the Fourier ellipsoid-bound info-gap model to quantify this fragmentary prior information.

For simplicity and clarity we will consider a single degree of freedom system with zero initial conditions. In the literature one finds many examples of multi-DOF analyses. The displacement $x(t)$ is described by the following linear differential equation:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = u(t), \quad x(0) = \dot{x}(0) = 0 \tag{5}$$

Let T be the duration of the transient, and express the input as the sum of a known part, $\widetilde{u}(t)$, and an unknown part expanded in harmonics:

$$u(t) = \widetilde{u}(t) + \sum_{n=1}^{N} [a_n \sin n\pi t/T + b_n \cos n\pi t/T] \tag{6}$$

This is more conveniently expressed in vector notation. Let $\beta^T = (a_1, \ldots, b_N)$ be the vector of unknown Fourier coefficients and let $\gamma(t)$ be the corresponding vector of sine and cosine functions, so that:

$$u(t) = \widetilde{u}(t) + \beta^T \gamma(t) \tag{7}$$

If we imagine a $2N$-dimensional space whose coordinates are the coefficients $\beta_1, \ldots, \beta_{2N}$, then the unknown vectors β cluster in a cloud in this space. Our prior information is sufficient to roughly characterize the shape of this cloud but not its size. This information about the uncertain coefficient vector β is represented by the following family of sets, which constitutes the Fourier ellipsoid-bound info-gap model:

$$\mathcal{U}(\alpha) = \left\{ \beta : \ \beta^T W \beta \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{8}$$

where W is a real, symmetric, positive definite matrix which specifies the shape of the ellipsoid, and the uncertainty parameter α determines the size of each ellipsoidal set in the family.

Let $\widetilde{x}(t)$ represent the displacement in response to the known part of the input, $\widetilde{u}(t)$. Eq.(5) can be solved (Inman, 1994) to express the total displacement at time t as:

$$x_u(t) = \widetilde{x}(t) + \beta^T \underbrace{\left[\frac{1}{m\omega_d} \int_0^t \gamma(\tau) \mathrm{e}^{-\xi\omega(t-\tau)} \sin \omega_d (t-\tau) \, \mathrm{d}\tau \right]}_{\zeta(t)} \tag{9}$$

$$= \widetilde{x}(t) + \beta^T \zeta(t) \tag{10}$$

where $\zeta(t)$ is defined in eq.(9), and is a known vector function. The natural frequency is $\omega = \sqrt{k/m}$, the damping coefficient is $\xi = c/2\omega m$, and the damped natural frequency is $\omega_d = \omega\sqrt{1-\xi^2}$. We have assumed that $\xi^2 < 1$.

It is now a matter of standard optimization theory to find the extremal displacements, subject to uncertainty in the Fourier coefficient vector β. That is, we must:

$$\text{Optimize} \ \ \beta^T \zeta(t), \qquad \text{Subject to} \ \ \beta^T W \beta \le \alpha^2 \tag{11}$$

$\beta^T \zeta(t)$ attains its extremal values at the boundary of the constraint because it is a linear function. The method of Lagrange optimization is directly applicable here, as is explained elsewhere (Ben-Haim and Elishakoff, 1990). The extremal responses are found to be:

$$\widehat{x}(t) = \widetilde{x}(t) \pm \alpha \sqrt{\zeta^T(t) W^{-1} \zeta(t)} \tag{12}$$

The coefficient vectors which result in these extremal responses at time t are:

$$\widehat{\beta} = \pm \frac{\alpha}{\sqrt{\zeta^T(t) W^{-1} \zeta(t)}} W^{-1} \zeta(t) \tag{13}$$

We notice that the extremal response $\widehat{x}$ depends on the uncertainty parameter α whose value is usually unknown. That is, the value of the extremal response is itself uncertain. This is a universal characteristic of info-gap analysis, and is central to the concept of robust reliability developed in section 4. ■

Example 2 *Maximum squared displacement with uncertain input.* In some applications it is not the displacement which is of interest, but its squared value. For instance, energy dissipation is related to quadratic functions of the displacement. We repeat the previous example, with the single modification that the nominal input $\widetilde{u}(t)$ is zero, implying that $\widetilde{x}$ also vanishes. Using eq.(10), the squared displacement can be expressed:

$$x^2(t) = \beta^T \zeta(t) \zeta^T(t) \beta \tag{14}$$

whose maximum we must seek subject to the constraint that β belongs to the info-gap model in eq.(8). Thus, we must:

$$\text{Maximize} \quad \beta^T \zeta(t) \zeta^T(t) \beta, \qquad \text{Subject to} \quad \beta^T W \beta \le \alpha^2 \tag{15}$$

This leads to an eigenvalue problem, whose solution is readily found using Lagrange optimization. Let $\mu(t)$ be the maximal eigenvalue of the real symmetric matrix $W^{-1/2}\zeta(t)\zeta^T(t)W^{-1/2}$. One then finds that the maximum squared displacement is:

$$\max x^2(t) = \alpha^2 \mu(t) \tag{16}$$

The Fourier coefficient vector β which produces this maximum squared displacement is the eigenvector of $W^{-1/2}\zeta(t)\zeta^T(t)W^{-1/2}$ corresponding to the maximum eigenvalue $\mu(t)$. ■

Example 3 *Hybrid probabilistic/info-gap uncertainty.* It often happens that some information about the uncertain phenomenon is probabilistic and some information is more amenable to info-gap representation. For example, in seismic excitation and other shock problems, we may know the probability density of the total energy of the event, while much less is known about the precise temporal waveform of the load on the structure. We will illustrate the integration of probabilistic and info-gap modelling with a simple example.

Let the cumulative probability of events whose total energy is no greater than E be a known function $P_E(E)$. Furthermore, for events at energy no greater than E, the uncertainty in the deviation of the input waveform $u(t)$ from the nominal input $\widetilde{u}(t)$, is represented by an energy-bound info-gap model like eq.(1):

$$\mathcal{U}(E, \widetilde{u}) = \left\{ u(t) : \int_0^\infty [u(t) - \widetilde{u}(t)]^2 \, \mathrm{d}t \le \kappa E \right\}, \quad E \ge 0 \tag{17}$$

where the uncertainty parameter is the total energy E, and κ is a constant which adjusts the units.

If the vibration is represented as a linear SDOF oscillator, the displacement in response to input $u(t)$, assuming sub-critical damping, is:

$$x_u(t) = \frac{1}{m\omega_d} \int_0^t u(\tau) \mathrm{e}^{-\xi\omega(t-\tau)} \sin \omega_d (t - \tau) \, \mathrm{d}\tau \tag{18}$$

where the parameters are defined as in connection with eq.(9).

The nominal input is $\widetilde{u}(t)$ which results in the nominal displacement $x_{\widetilde{u}}(t)$. Employing the Schwarz inequality (Ben-Haim and Elishakoff, 1990) one can find the greatest deviation of the displacement from the nominal value, for any input up to energy E:

$$\left| x_u(t) - x_{\widetilde{u}}(t) \right| \le \eta(t) \sqrt{\kappa E} \quad \text{for all} \quad u(t) \in \mathcal{U}(E, \widetilde{u}) \tag{19}$$

where we have defined:

$$\eta(t) = \frac{1}{m\omega_d} \sqrt{\int_0^t \mathrm{e}^{-2\xi\omega\tau} \sin^2 \omega_d \tau \, \mathrm{d}\tau} \tag{20}$$

Let us suppose that significant damage occurs if the displacement $x(t)$ exceeds a critical value $x_{\rm cr}$. That is, failure is defined as:

$$x(t) > x_{\rm cr} \tag{21}$$

The classical probabilistic reliability is the probability of no-failure, which is $\text{Prob}(x_u \le x_{\rm cr})$. We are not able to derive an exact expression for this probability since part of the uncertainty — the unknown input waveform $u(t)$ — has eluded a probabilistic representation.

Employing the cumulative probability distribution of the input energy, $P_E(E)$, and inequality (19) based on the info-gap model, we obtain the following bound on the probability that the displacement does not exceed the critical value:

$$\text{Prob}(x_u \le x_{\rm cr}) \ge P_E \left[\frac{1}{\kappa} \left(\frac{x_{\rm cr} - x_{\widetilde{u}}}{\eta} \right)^2 \right] \tag{22}$$

We are able to obtain no more than a bound on the probability of avoiding significant damage, because we have only info-gap information about the temporal load uncertainty. However, we are able to exploit the partial probabilistic information embodied in the energy distribution $P_E(E)$. ■

4 Robust Reliability

We now come to the main applicative part of this paper: evaluating the reliability of a vibrating mechanical system when the the inputs are very poorly known. Since we have, at best, only partial probabilistic information, we will seek a supplement to classical probabilistic reliability.

To *rely* on something means to have confidence based on experience. This is a plain English word, and it has carried this meaning since long before engineers started thinking probabilistically. Reliability rests on two more primitive concepts: performance and uncertainty. Still speaking lexically and not technically, we can rely on something when, despite uncertainties, its performance will be acceptable.

We now ask for a quantitative theory which reflects the intuitive idea of reliability. The current standard theory of reliability is based on probability: the reliability of a system is measured by the probability of no-failure. This approach is exceedingly useful and has been developed in recent decades by many able authors.

In this paper we will describe a different formulation. We measure the reliability of a system by *the amount of uncertainty consistent with no-failure.* That is, reliability can be quantified as *immunity to uncertainty.* A reliable system will perform satisfactorily in the presence of great uncertainty and is immune to unanticipated variations. Such a system is robust with respect to uncertainty, and hence the name *robust reliability.* On the other hand, a system has low reliability when small fluctuations can lead to failure. Such a system is fragile or vulnerable to uncertainty. This approach to reliability is described elsewhere in detail (Ben-Haim, 1994c, 1995, 1996, 1997a). We will illustrate the method of robust reliability with several examples.

We will employ three components in our analysis of the robust reliability of mechanical systems: (1) a model of the mechanics, (2) a model of the uncertainties, and (3) a criterion of failure. For items (1) and (3) we will use standard mechanical and physical theories. For item (2) we will use info-gap models of uncertainty, augmented where possible with whatever probabilistic information is available.

Example 4 *Reliability of an SDOF system with a potential-energy failure criterion.* Consider a linearly vibrating SDOF system subject to uncertain input. The **mechanical model** is eq.(5). The **input uncertainty model** is the info-gap model $\mathcal{U}(\alpha, \widetilde{u})$, $\alpha \ge 0$, which is made up of eqs.(7) and (8). The system fails if the potential energy $kx^2/2$ stored in the system exceeds a critical value $E_{\rm cr}$. The **failure criterion** can be expressed:

$$x^2(t) > \frac{2E_{\rm cr}}{k} \tag{23}$$

where k is the stiffness of the SDOF system.

The robust reliability — measured as the degree of the immunity of the system to uncertainty — is the greatest value of the uncertainty parameter α consistent with no-failure of the system. Formally we can define the robust reliability $\widehat{\alpha}$ as follows:

$$\widehat{\alpha} = \max \left\{ \alpha : \ x_u^2(t) \le \frac{2E_{\rm cr}}{k}, \quad \text{for all} \quad u(t) \in \mathcal{U}(\alpha, \widetilde{u}) \right\} \tag{24}$$

We can "read" this relation as: the robust reliability $\widehat{\alpha}$ is the maximum of the set of α-values for which the squared displacement $x_u^2(t)$ in response to input $u(t)$ is no greater than $2E_{\rm cr}/k$ for all functions $u(t)$ in the input uncertainty-set $\mathcal{U}(\alpha, \widetilde{u})$.

From eq.(16) we know that the greatest squared displacement, for any input up to uncertainty α, is $\alpha^2\mu(t)$, where μ is the maximum eigenvalue defined in example 2. The robust reliability is evaluated by equating the maximum squared displacement to the threshold value $2E_{\rm cr}/k$ and then solving for the uncertainty parameter:

$$\max_{u \in \mathcal{U}(\alpha, \widetilde{u})} x_u^2 = \frac{2E_{\rm cr}}{k} \quad \Longrightarrow \quad \widehat{\alpha} = \sqrt{\frac{2E_{\rm cr}}{k\mu(t)}} \tag{25}$$

$\widehat{\alpha}$ is the greatest value of the uncertainty parameter consistent with no-failure. If $\widehat{\alpha}$ is large, then the system is relatively immune to uncertainty, while if $\widehat{\alpha}$ is small, then the system is vulnerable to uncertain variation of the input and cannot be relied upon to perform its mission.

An important question in any reliability analysis — probabilistic as well as robust reliability — is the subjective calibration of the reliability index. In the present case: how robust is robust enough? Or, how large a value of $\widehat{\alpha}$ is 'large'? This can be addressed in various ways (Ben-Haim, 1996, ch. 9). A simple and often accessible approach is to 'calibrate' the robustness by some non-dimensional normalization. For instance, suppose that we know a typical length of oscillation, $\widetilde{x}$. Then a typical force of displacement of the system is $k\widetilde{x}$. Since $\widehat{\alpha}$ has units of force (to match the units of the input function $u(t)$) we can calibrate the robust reliability by normalizing it with $k\widetilde{x}$. That is, $\widehat{\alpha}/k\widetilde{x}$ is dimensionless, so if $\widehat{\alpha}/k\widetilde{x} \ll 1$, then the system is very vulnerable to uncertainty in the input, since force fluctuations much less than typical resistence forces entail failure. On the other hand, $\widehat{\alpha}/k\widetilde{x} \gg 1$ implies substantial robustness because force fluctuations much larger than the typical stiffness force do not lead to failure. ■

Example 5 *Reliability with hybrid probabilistic/info-gap uncertainty.* Consider a damped SDOF mechanical system subject to separate transient excitations of duration T occurring infrequently enough so that the system damps down to zero initial conditions after each event. The **mechanical model** for each transient is eq.(5).

Based on prior information, the probability of exactly n transients in a long duration Θ is described by the Poisson probability distribution:

$$P_n(\Theta) = \frac{\mathrm{e}^{-\lambda\Theta}(\lambda\Theta)^n}{n!}, \quad n = 0,\, 1,\, 2,\, \ldots \tag{26}$$

where λ is the mean number of transients per unit time. However, the specific temporal variation of each transient input is uncertain and described by the info-gap model of eqs.(7) and (8) with $\widetilde{u} = 0$. Thus the **uncertainty model** combines both probabilistic and info-gap models of uncertainty, $P_n(\Theta)$ and $\mathcal{U}(\alpha)$.

The system fails when damage has accumulated in excess of a quantity $E_{\rm cr}$. While there are many models of damage accumulation, we adopt the simplest for purposes of illustration. Extensions can be found in Ben-Haim (1994b, 1996). We suppose that the increment of damage in a single transient is proportional to the integral squared displacement:

$$E = \nu \int_0^T x_u^2 \, \mathrm{d}t \tag{27}$$

where ν is a constant. Failure occurs in n transients with damage increments $E_1, \ldots, E_n$ if:

$$\sum_{i=1}^{n} E_i > E_{\rm cr} \tag{28}$$

This is the **failure criterion.**

As in example 2, one can show that the maximum increment of damage in a single transient is:

$$\max_{u \in \mathcal{U}(\alpha)} E = \nu \max_{u \in \mathcal{U}(\alpha)} \int_0^T x_u^2 \, \mathrm{d}t \tag{29}$$

$$= \nu \alpha^2 \eta \tag{30}$$

where η is the greatest eigenvalue of the matrix $W^{-1/2} \int_0^T \zeta^T(t)\zeta(t) \, \mathrm{d}t \, W^{-1/2}$ and ζ is defined in eq.(9).

We have insufficient probabilistic information to calculate the probability of failure: we can not calculate the probability distribution of $\max \int x_u^2 \, \mathrm{d}t$ since u is characterized only by an info-gap model of uncertainty. However, we have partial probabilistic information: the distribution in time of the transients is Poisson. A hybrid approach is called for, and we will outline one such analysis (Ben-Haim, 1996, pp.193–194).

The robust reliability for failure in exactly n transients is the greatest value of the uncertainty parameter α which is consistent with no-failure. Call this $\widehat{\alpha}_n$, which can be evaluated based on the info-gap model. Since we know the probability of n transients in duration Θ, we can average $\widehat{\alpha}_n$ on the number of transients.

Combining eqs.(27)–(30), we evaluate $\widehat{\alpha}_n$ by equating the maximum cumulative energy to the threshold value, and then solving for α:

$$n\nu\alpha^2\eta = E_{\rm cr} \quad \Longrightarrow \quad \widehat{\alpha}_n = \sqrt{\frac{E_{\rm cr}}{n\nu\eta}} \tag{31}$$

This is the robust reliability for failure in precisely n transient events.

Randomizing $\widehat{\alpha}_n$ on n, we must exclude the case $n = 0$ since failure can not occur without the input of energy. Hence the robust reliability averaged over the Poisson probability distribution becomes:

$$\widehat{\alpha} = \frac{1}{1 - P_0(\Theta)} \sum_{n=1}^{\infty} \widehat{\alpha}_n P_n(\Theta) \tag{32}$$

$$= \frac{\mathrm{e}^{-\lambda\Theta}}{1 - \mathrm{e}^{-\lambda\Theta}} \sqrt{\frac{E_{\rm cr}}{\nu\eta}} \sum_{n=1}^{\infty} \frac{(\lambda\Theta)^n}{n!\sqrt{n}} \tag{33}$$

Eq.(33) reveals how the reliability decreases as the duration of operation, Θ, increases. It shows the dependence of the reliability on the failure threshold $E_{\rm cr}$ and on the system properties through the quantities ν and η. ■

5 Risk Aversion and Engineering Decisions

The analysis of reliability serves the engineering decision-maker in various fields such as safety analysis and certification, design of systems and processes, monitoring and fault diagnosis, and so on. In this section we introduce the idea of risk aversion, borrowed to some extent from economic theory, and show how it is used, together with robust reliability, as a decision-support tool in vibration problems with uncertainty. We will show that the robustness $\widehat{\alpha}$, viewed as a function of the failure threshold, gives a quantitative global characterization of the system and its relation to its uncertain environment.

To begin with an example, note that in eq.(25) the robust reliability $\widehat{\alpha}$ increases monotonically with the failure threshold $E_{\rm cr}$. This is true quite generally, and provides the basis of our discussion.

It is intuitively quite reasonable that the reliability increases with the failure threshold of the system. We will obtain deeper insight into the theory of robust reliability by seeking to understand the origin of the universal monotonic relation between reliability and failure threshold. First we need a more general formulation of robust reliability. Let the state of the system be $x(t)$ and its input be $u(t)$, which are related by the **mechanical model,** whatever it may be. The **failure criterion** states that failure occurs if a 'decision function' $D(x,u,t)$ exceeds a threshold value $D_{\rm cr}$:

$$D(x,u,t) > D_{\rm cr} \tag{34}$$

where x obeys the dictates of the mechanical model of the system. This encompasses quite a broad range of failure criteria, including those we have considered, eqs.(21), (23) and (28). The **uncertainty model** is an info-gap model, $\mathcal{U}(\alpha,\widetilde{u})$, $\alpha \ge 0$.

To formulate the robust reliability of the system we first need to define the following set:

$$\mathcal{A}(D_{\rm cr}) = \{\alpha : \ D(x,u,t) \le D_{\rm cr}, \quad \text{for all} \quad u \in \mathcal{U}(\alpha,\widetilde{u})\} \tag{35}$$

$\mathcal{A}(D_{\rm cr})$ is the set of all values of the uncertainty parameter α for which the decision function $D(x,u,t)$ does not violate the failure criterion, for all inputs $u(t)$ in the info-gap model $\mathcal{U}(\alpha,\widetilde{u})$. The robust reliability is the greatest α in this set:

$$\widehat{\alpha}(D_{\rm cr}) = \max \mathcal{A}(D_{\rm cr}) \tag{36}$$

In other words, as we have stated before, the robust reliability is the greatest value of the uncertainty parameter which is consistent with no-failure of the system. Eq.(24) is a specific example of this procedure.

Without getting into mathematical technicalities, one can see that, if failure threshold D_1 is less than threshold D_2, then $\mathcal{A}(D_1)$ is a subset of $\mathcal{A}(D_2)$:

$$D_1 < D_2 \quad \Longrightarrow \quad \mathcal{A}(D_1) \subseteq \mathcal{A}(D_2) \tag{37}$$

The explanation: if $D(x,u,t) \le D_1$ for all u in $\mathcal{U}(\alpha,\widetilde{u})$, then clearly $D(x,u,t) \le D_2$ for all u in $\mathcal{U}(\alpha,\widetilde{u})$ since $D_1 < D_2$. Hence any α in $\mathcal{A}(D_1)$ will also belong to $\mathcal{A}(D_2)$.

As a start at understanding what this means, let's state this result succintly and give it a name:

Gambler's theorem: The robust reliability increases monotonically with the failure threshold.

The gambler's theorem can be interpreted in various ways in different contexts. We will examine it in connection with design decisions. Let us return to eq.(25) in example 4. In fig. 1 we plot the robustness versus the failure threshold for two different designs, for instance, provided by designers who have chosen different natural frequencies. The robustness axis is labelled 'immune'

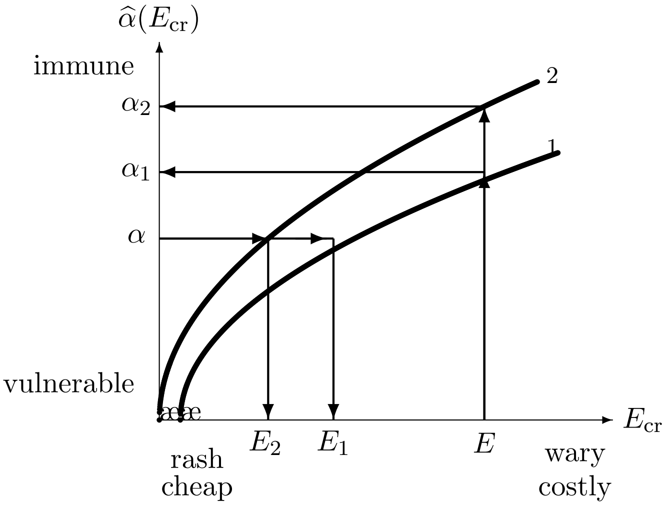

Figure 1: Robustness versus failure threshold for two different designs.

for large values of $\widehat{\alpha}$ since large robustness implies large immunity to uncertainty. Likewise, the $\widehat{\alpha}$ axis is labelled 'vulnerable' for small values of $\widehat{\alpha}$. Consider now the failure-threshold axis. When the designer chooses his system to fail at a low threshold, he is adopting an approach which might be viewed, in the extreme, as 'rash' or 'cavalier', motivated perhaps by the desire for an inexpensive product. At the other extreme, where $E_{\rm cr}$ is very large, the designer may choose expensive solutions providing high failure thresholds. With this point of view in mind we have labelled the failure-threshold axis 'rash' or 'cheap' for small values of $E_{\rm cr}$ and 'wary' or 'costly' for large $E_{\rm cr}$.

With this interpretation of the axes in fig. 1, we can interpret the monotonic increase of the robustness curve as expressing the trade-off between cost and vulnerability-to-uncertainty: as cost rises, vulnerability decreases. Conversely, the gambler's theorem states the trade-off between economy and immunity-to-uncertainty: as the designer "cuts corners and costs", he increases his vulnerability to uncertainty.

Furthermore, the robustness curve quantitatively characterizes the designer in terms of how he "gambles" in this trade-off. Comparing the two designs in the figure we see that designer #2 shows greater fondness for risk and uncertainty than designer #1. Consider the arrows running to the right and downward from the robustness value α. At the level α of ambient uncertainty, designer #1 is more cautious and requires a more costly system than designer #2, as evidenced by the fact that $E_1 > E_2$. In this sense, designer #1 is more averse to risk than designer #2. Likewise, consider the arrows rising from the failure-threshold E. For a system intended to fail at this threshold, designer #2 can tolerate greater ambient uncertainty than designer #1, since $\alpha_2 > \alpha_1$. Again, designer #2 has greater proclivity for ambient uncertainty than designer #1.

Whatever linguistic interpretation one chooses for the robustness curve, and it must be stressed that the interpretation of the last few paragraphs is by no means the only plausible one, the fact remains that robustness increases monotonically with failure threshold. The gambler's theorem expresses a trade-off between vulnerabilty-to-uncertainty and resistence-to-failure: vulnerability increases as resistence falls. Robustness curves such as those in fig. 1 serve as an assessment of global sensitivity to uncertainty.

6 Summary

We have addressed the challenge of using imperfect information about dynamic loads in evaluating the reliability of a vibrating system. Uncertain inputs can be characterized in different ways, using probability theory, or information-gap models of uncertainty, or fuzzy logic. These methods are conceptually distinct and each is particularly suited to a different type and quantity of prior information.

We have stressed info-gap models of uncertainty, which are adapted to severe lack of information. In situations where rare events are of critical concern and information is at a premium, info-gap models allow the analyst to make rational and systematic assessments without introducing unverifiable extrapolations beyond his data. When some probabilistic information is available, we have shown how it can be incorporated in a hybrid probabilistic/info-gap analysis.

We have outlined the theory of robust reliability, which replaces probabilistic reliability in those situations where prior information is insufficient to verify the choice of a probability density. Here also we have illustrated a hybrid probabilistic/info-gap reliability analysis.

Finally, we used the idea of aversion to risk to provide an overall assessment of the performance of a system in an uncertain environment. The "gambler's theorem" shows that the robustness $\widehat{\alpha}$, viewed as a function of the failure threshold $E_{\rm cr}$, quantifies the trade-off between inherent system properties and environmental uncertainty.

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