# The Equity Premium: A Solution

Yakov Ben-Haim
Technion — Israel Institute of Technology
Haifa 32000 Israel
yakov@technion.ac.il

**Abstract**

An unconventional decision model describes consumption and investment with severe Knightian uncertainty. Investors do not maximize utility. Instead, utility is satisficed and robustness to uncertainty in returns is maximized. Information-gap models quantify Knightian uncertainty. Discounted life-time utility leads to an info-gap generalization of the Lucas asset-pricing model. This suggests an explanation of the equity premium puzzle without large Arrow-Pratt risk aversion. With robust-satisficing, we show that reasonable values of risk aversion and discount rate are consistent with observed equity premium, risk-free rate and consumption growth.

**JEL Classification:** G11 (Portfolio Choice; Investment Decisions), G12 (Asset Pricing).

## Contents



[0]\papers\equity_pzl\eqp\ta09.tex 1.10.2005. 

# 1 Introduction

This paper presents an unconventional decision model for relating consumption to the returns on both highly uncertain and risk-free investments. This decision model provides a solution to the equity premium puzzle.

Rather than assuming that investors choose the investment so as to maximize the total discounted utility, we assume they **satisfice the utility** and **maximize the robustness against uncertainty** in the future returns. We refer to this decision strategy as 'robust-satisficing'. Investors cannot confidently maximize utility because of the severe uncertainty in asset returns. The challenge facing investors is to decide whether adequate returns are sufficiently reliable. If not, then the resources can be invested elsewhere.

The crux of the matter is the treatment of uncertainty. We use an **information-gap model** to represent the investor's Knightian uncertainty of future returns (Ben-Haim, 2001): an unmeasurable and non-probabilistic epistemic gap between known past returns and unknown future returns. The info-gap formulation presumes that the investor knows the past returns, believes that future returns may deviate greatly from past experience, and that reliable probabilistic models of these deviations are unavailable. An info-gap model uses an unbounded family of nested sets to represent possible payoffs. No measure functions are involved.

The investor has little confidence that past returns reflect future behavior reliably. The investor's central robustness question, which motivates our decision model, is: how wrong can the current estimate of future returns be, without jeopardizing the attainment of a specified level of utility? The answer to this question generates preferences on options, without requiring probabilistic information.

We present a multi-period two-asset implementation which provides insight into the equity premium puzzle. In section 2 we formulate the investment model, the info-gap model of uncertainty in future returns, the robustness function and the robust-satisficing decision strategy. In section 3 we derive asset-pricing relations which are the info-gap generalization of the Lucas asset-pricing model. In section 4 we derive an expression for the equity premium and discuss robustness, consumption growth, and risk aversion. In section 5 we discuss an info-gap concept of stationarity. A methodological summary and discussion of the results are presented in section 6.

# 2 Dynamics, Uncertainty and Robustness

We consider discrete time, $t = 0, 1, \ldots, T$. $c_t$ is the total consumption at time step $t$ and $u(c_t)$ is the utility from this consumption. We assume that $u(c_t)$ is continuous and that the marginal utility is positive: $u'(c_t) > 0$.

We consider two assets, one a risky stock ($i = 1$) and the other a risk-free bond ($i = 2$). The generalization to more than two assets is straightforward and would not entail any substantive alteration of our conclusions.

$x_{it}$ is the quantity of asset $i$ held between $t$ and $t+1$ by a representative agent and can be either positive or negative. The holdings at time $t$ are $x_t = (x_{1t}, x_{2t})'$. The holdings throughout the time horizon are $x = (x_0, x_1, \ldots, x_{T-1})$. $x$ is chosen by the investor.

$p_{it}$ is the ex-dividend price of asset $i$ at time $t$, where $p_t = (p_{1t}, p_{2t})'$. $d_{it}$ is the dividend of asset $i$ at time $t$. The prices throughout the time horizon are $p = (p_0, p_1, \ldots, p_{T-1})$.

$y_{it} = p_{it} + d_{it}$ is the payoff of asset $i$ at time $t$, where $y_t = (y_{1t}, y_{2t})'$. $y_0$ is known, $y_{1t}$ is risky and uncertain while $y_{2t}$ is risk-free and known at $t = 0$ for all $t \geq 0$. The uncertain payoffs are $y_{(1)} = (y_{11}, \ldots, y_{1T})$. $y_{1t}$ can be either positive or negative. (Requiring non-negative payoff would alter our analysis only at robustness in excess of 100% of anticipated returns, which we will see is not of practical interest for understanding the equity premium puzzle.)

The budget constraint is:

$$c_t + p_t' x_t = y_t' x_{t-1}, \quad t = 0, \ldots, T \tag{1}$$

The initial endowment, $x_{-1}$ and $y_0$, are known at time 0. The investment in the last step is zero: $x_T = 0$. Short sells are allowed but $c_t$ cannot be negative. The choice variables are $x_0, \ldots, x_{T-1}$, which determine the consumptions through the budget constraints.

The payoff of the risky asset, $y_{1t}$, is uncertain for $t > 0$, and $\widetilde{y}_{1t}$ is the best known estimate of $y_{1t}$ at time 0. We assume that the anticipated payoff is positive: $\widetilde{y}_{1t} > 0$. $y_{10}$ is known. The investor views $\widetilde{y}_{1t}$ as a best but rough estimate of future payoffs, without knowing how wrong this estimate will turn out to be. We use the unbounded fractional-error info-gap model to represent uncertainty in the risky-asset payoffs:

$$\mathcal{Y}(\alpha, \widetilde{y}) = \left\{ y_{(1)} : \ |y_{1t} - \widetilde{y}_{1t}| \le \alpha \widetilde{y}_{1t}, \quad t = 1, \ldots, T \right\}, \quad \alpha \ge 0 \tag{2}$$

$\mathcal{Y}(\alpha, \widetilde{y})$ is the set of risky payoffs $y_{1t}$, for $t = 1, \ldots, T$, whose fractional deviations from the anticipated payoffs $\widetilde{y}_{1t}$ are no greater than $\alpha$. The fractional error, $\alpha$, is unknown, so the 'horizon of uncertainty' is unbounded. This info-gap model is not a single interval, but rather an unbounded family of nested payoff intervals.

The discounted utility up to time $T$ is:

$$U(x, y) = \sum_{t=0}^{T} \beta^t u(c_t) \tag{3}$$

The investor desires to choose the asset holdings $x$ throughout the time horizon so as to attain discounted utility no less than $\overline{U}$. That is, the investor wishes to satisfice the discounted utility at the value $\overline{U}$:

$$U(x, y) \ge \overline{U} \tag{4}$$

$\overline{U}$ can be thought of as a 'reservation utility'. The investment will be pursued if the investor has adequate confidence in achieving adequate reward.

The **robustness** to uncertainty in the payoffs, of holdings $x$ with utility-aspiration $\overline{U}$, is the greatest horizon of uncertainty $\alpha$ up to which all payoffs yield at least the desired utility:

$$\widehat{\alpha}(x, \overline{U}) = \max \left\{ \alpha : \ \left( \min_{y_{(1)} \in \mathcal{Y}(\alpha, \widetilde{y})} U(x, y) \right) \ge \overline{U} \right\} \tag{5}$$

The set of $\alpha$-values in this definition is empty if $U(x, \widetilde{y}) < \overline{U}$, meaning that holdings $x$ do not attain utility $\overline{U}$ with the anticipated payoffs $\widetilde{y}$. In this case we define $\widehat{\alpha}(x, \overline{U}) = 0$ and we say that utility aspiration $\overline{U}$ is 'infeasible'. Any other $\overline{U}$ is 'feasible'.

We will be interested in robust-satisficing investments: those which satisfice the discounted utility and maximize the robustness:

$$\widehat{x}(\overline{U}) = \arg\max_{x} \widehat{\alpha}(x, \overline{U}) \tag{6}$$

where the maximum on $x$ is subject to the budget constraint with non-negative consumption.

# 3 Asset-Pricing Relation

Consider the time horizon $t = 0, 1, \ldots, T$. The investor has $T$ choice vectors $x_0 = (x_{10}, x_{20})', \ldots, x_{T-1} = (x_{1,T-1}, x_{2,T-1})'$. The uncertainties are the unknown risky payoffs $y_{(1)} = (y_{11}, \ldots, y_{1T})'$. The info-gaps in these payoffs are described by $\mathcal{Y}(\alpha, \widetilde{y})$ in eq.(2) where the horizon of uncertainty $\alpha$ is unknown. The risk-free payoffs $(y_{20}, \ldots, y_{2T})$ are known. Using the budget constraint in eq.(1), the discounted utility in eq.(3) is:

$$U(x, y) = \sum_{t=0}^{T} \beta^t u(y_{1t} x_{1,t-1} + y_{2t} x_{2,t-1} - p_t' x_t) \tag{7}$$

$x_{1t}$ can be either positive or negative. Let $\phi_t = 1$ if $x_{1t} \ge 0$ and $\phi_t = -1$ otherwise for $t \ge 0$. Define $\phi_{-1} = 0$. The marginal utility is positive, so the lowest utility up to horizon of uncertainty $\alpha$

occurs when the risky payoffs $y_{1t}$ are such that $y_{1t}x_{1,t-1}$ is minimal, for $t = 1, \ldots, T$. (Recall that $y_{10}$ is known.) The risky payoffs which minimize the utility at uncertainty $\alpha$ are $y_{1t} = \widetilde{y}_{1t}(1-\alpha\phi_{t-1})$. Thus the minimum in the definition of the robustness, eq.(5), is:

$$
\begin{aligned}
\mu(\alpha, x, p) &= \min_{y_{(1)} \in \mathcal{Y}(\alpha, \widetilde{y})} U(x, y) & (8) \\
&= \sum_{t=0}^{T} \beta^t u[\underbrace{(1-\alpha\phi_{t-1})\widetilde{y}_{1t}x_{1,t-1} + y_{2t}x_{2,t-1} - p_t' x_t}_{\widetilde{c}_t(\alpha)}] & (9)
\end{aligned}
$$

which defines $\widetilde{c}_t(\alpha)$, the lowest anticipated consumption up to uncertainty $\alpha$.

The utility aspiration $\overline{U}$ is feasible if the righthand side of eq.(9) is no less than $\overline{U}$ in the absence of uncertainty ($\alpha = 0$). Because the marginal utility is positive and $\alpha\phi_{t-1}\widetilde{y}_{1t}x_{1,t-1}$ is also positive, the righthand side of eq.(9) decreases strictly monotonically as $\alpha$ increases. Hence, for any feasible aspiration, the robustness is the value of $\alpha$ at which the righthand side of eq.(9) equals $\overline{U}$:

$$
u(\underbrace{y_0' x_{-1} - p_0' x_0}_{c_0}) + \sum_{t=1}^{T} \beta^t u \left[ \underbrace{\left(1 - \widehat{\alpha}(x, \overline{U})\phi_{t-1}\right) \widetilde{y}_{1t}x_{1,t-1} + y_{2t}x_{2,t-1} - p_t' x_t}_{\widetilde{c}_t(\widehat{\alpha})} \right] = \overline{U} \qquad (10)
$$

If eq.(10) holds for a continuum of investments $x$, then its derivatives with respect to $x_{1t}$ and $x_{2t}$ also hold. Suppose there is a robust-satisficing investment which maximizes the robustness without constraint (we study the existence of such investments in section 5):

$$
\frac{\partial \widehat{\alpha}}{\partial x_{it}} = 0, \quad i = 1, 2; \; t = 0, \ldots, T-1 \qquad (11)
$$

Differentiating eq.(10) with respect to $x_{1t}$ (which we assume differs from zero) and $x_{2t}$, for $t = 0, \ldots, T-1$, and using eq.(11) yields:

$$
\begin{aligned}
\frac{\mathrm{d}u(c_t)}{\mathrm{d}c_t} p_{1t} &= \beta \frac{\mathrm{d}u(c_{t+1})}{\mathrm{d}c_{t+1}} (1 - \widehat{\alpha}\phi_t)\widetilde{y}_{1,t+1} & (12) \\
\frac{\mathrm{d}u(c_t)}{\mathrm{d}c_t} p_{2t} &= \beta \frac{\mathrm{d}u(c_{t+1})}{\mathrm{d}c_{t+1}} y_{2,t+1} & (13)
\end{aligned}
$$

where $c_t = \widetilde{c}_t(\widehat{\alpha})$ is the argument of the utility function in eq.(10). These relations are the info-gap generalizations of the first-order conditions in the Lucas asset-pricing model (see Blanchard and Fischer, 1989, eq.(11), p.511). The ordinary Lucas relations result when $\widehat{\alpha} = 0$.

Eq.(12) asserts that the risky-asset price $p_{1t}$ depends not only on the marginal rate of substitution as in the Lucas model, but also depends on the required robustness $\widehat{\alpha}$. At low risk aversion the marginal utility is nearly constant, so the dominant term on the righthand of eq.(12) is $1 - \widehat{\alpha}\phi_t$. When $\phi_t = +1$ (e.g. the representative agent has positive risky holdings), this means that the risky-asset price tends to decrease as the robustness which the investor requires increases. On the other hand, for short sales of the risky asset (so $\phi_t = -1$), the price tends to increase as the robustness which the investor requires increases.

A basic theorem of info-gap theory asserts that robustness decreases as aspiration increases: $\widehat{\alpha}(x, \overline{U})$ decreases as $\overline{U}$ increases. This holds both for arbitrary investments $x$ and for the robust-satisficing investments $\widehat{x}(\overline{U})$ in eq.(6). Furthermore, the robustness vanishes at the greatest feasible aspiration: $\widehat{\alpha}(x, \overline{U}) = 0$ if $\overline{U} = U(x, \widetilde{y})$ (Ben-Haim, 2001, 2005). (These results derive from the nested structure of info-gap models, e.g. $\mathcal{Y}(\alpha, \widetilde{y})$ in eq.(2), and the definition of the robustness, eq.(5).) This trade-off between robustness and utility-aspiration is illustrated schematically in fig. 1. Fig. 2 illustrates eq.(12) for low risk aversion and $\phi_t = +1$. Consequently, combining figs. 1 and 2, the risky-asset price rises as investor aspirations increase as shown in fig. 3.

The solid dots in figs. 1–3 identify corresponding points. In fig. 1 the robustness vanishes at the maximal utility aspiration. In fig. 2 we see that zero robustness corresponds to maximal risky-asset price, which is the analog of the Lucas asset price. $p_{10}^{\mathrm{c}}$ represents the price when the aspiration is maximal and the robustness is zero. In fig. 3 this maximal price corresponds to maximal utility aspiration. Sub-maximal utility-aspiration (equivalently: positive robustness-aspiration) will force a drop in the price of risky assets held by the representative agent ($\phi_t = +1$).

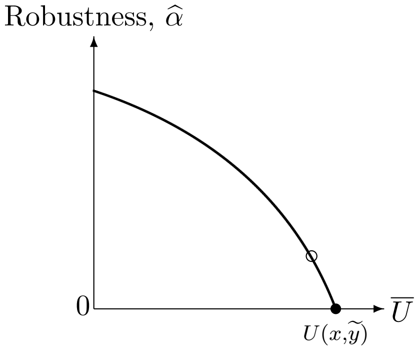


Figure 1: Trade-off of robustness $\widehat{\alpha}(x, \overline{U})$ against utility aspiration $\overline{U}$.

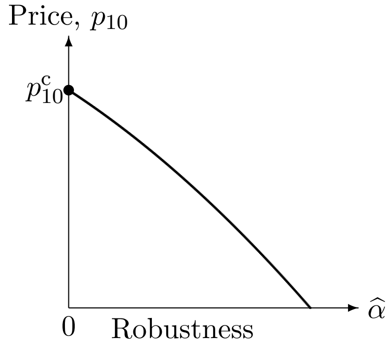


Figure 2: Price of risky asset vs. robustness, eq.(12) for $\phi_t = +1$.

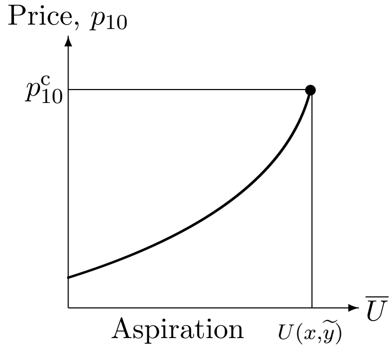


Figure 3: Price of risky asset vs. utility aspiration for $\phi_t = +1$.

# 4 Equity Premium

Define the anticipated or 'best-estimate' rates of return to the risky asset: $\widetilde{r}_{1t} = \widetilde{y}_{1t}/p_{1,t-1}$, for $t = 1, \ldots, T$. Likewise the known rates of return to the risk-free asset are $r_{2t} = y_{2t}/p_{2,t-1}$. With these definitions we can write eqs.(12) and (13), for $t = 0, \ldots, T-1$, as:

$$u'(c_t) = \beta u'(c_{t+1})(1 - \widehat{\alpha}\phi_t)\widetilde{r}_{1,t+1} \qquad (14)$$

$$u'(c_t) = \beta u'(c_{t+1}) r_{2,t+1} \qquad (15)$$

We now use these relations to show, in the three different ways, how the info-gap robust-satisficing decision paradigm provides insight into the equity premium puzzle.

**Robustness.** Subtracting eq.(15) from eq.(14) results in the info-gap generalization of a basic CAPM relation (see Blanchard and Fischer, 1989, eq.(4), p.507):

$$0 = \beta u'(c_{t+1})[(1 - \widehat{\alpha}\phi_t)\widetilde{r}_{1,t+1} - r_{2,t+1}], \quad t = 0, \ldots, T-1 \qquad (16)$$

This relation can be re-arranged (assuming $\beta u'(c_{t+1}) > 0$) to show that the premium for the risky asset is proportional to the required robustness:

$$\widetilde{r}_{1,t+1} - r_{2,t+1} = \widehat{\alpha}\, \phi_t \widetilde{r}_{1,t+1} \qquad (17)$$

Investors who do not require robustness to uncertainty ($\widehat{\alpha} = 0$) also do not need a premium to attract them to have positive holdings on risky assets ($\phi_t = +1$). As the investor becomes more sensitive to Knightian uncertainty in the returns (as $\widehat{\alpha}$ increases), a greater premium is needed to induce investment in both risky and risk-free assets.

Investors need not be terribly sensitive to Knightian uncertainty in order to explain the usual equity premium. For example, if $\widetilde{r}_{1,t+1} = 1.07$ and $r_{2,t+1} = 1.01$, then the robustness in eq.(17) which explains this 6% premium is $\widehat{\alpha} = 0.06/1.07 \approx 0.056$. This is fairly low robustness compared to the variation of risky returns which is on the order of 15%. In light of the monotonic trade-off between robustness and aspiration for utility, illustrated in fig. 1, this low robustness suggests that investors reduce their aspirations only slightly below the maximum. This is illustrated schematically by the open circle on the robustness curve of fig. 1. This small reduction in aspiration, in response to Knightian uncertainty in the returns, is responsible for the observed premium for the risky asset.

Note that the only assumptions concerning the utility function $u(c)$ are that it is continuous and that the marginal utility is positive. We have assumed nothing about the magnitude of Arrow-Pratt risk aversion. We have assumed time-separation of the discounted utility $U(x, y)$. The time horizon $T$ is arbitrarily large.

**Consumption growth.** The equity premium can be understood in another manner from eqs.(14) and (15), with explicit reference to consumption growth. Define $m = \beta u'(c_{t+1})/u'(c_t)$. In stochastic pricing models the expectation of this quantity is sometimes called the stochastic discount factor (Cochrane, 2001). Now eqs.(14) and (15) can be combined to yield the equity premium as:

$$(\widetilde{r}_{1,t+1} - r_{2,t+1})m = \frac{\widehat{\alpha}\phi_t}{1 - \widehat{\alpha}\phi_t} \tag{18}$$

Consumption growth and equity premium imply robustness. If, as before, $\phi_t = +1$, $\widetilde{r}_{1,t+1} - r_{2,t+1} = 0.06$ and $m = 1/1.01$, then the robustness required by investors is again about 0.056.

**Risk aversion.** Eq.(14) can be written as:

$$1 = (1 - \widehat{\alpha}\phi_t)m\widetilde{r}_{1,t+1} \tag{19}$$

where as before $m = \beta u'(c_{t+1})/u'(c_t)$. Both $\widetilde{r}_{1,t+1}$ and $m$ depend on anticipations based on information available at time $t = 0$. At time 0 the investor could roughly estimate these future values as historical expectations. Specifically:

$$\mathrm{E}_0(m\widetilde{r}_{1,t+1}) = \mathrm{E}_0(m)\mathrm{E}_0(\widetilde{r}_{1,t+1}) + \sigma_0(m)\sigma_0(\widetilde{r}_{1,t+1})\rho_0(m,\, \widetilde{r}_{1,t+1}) \tag{20}$$

where $\rho_0(m,\, \widetilde{r}_{1,t+1})$ is the correlation coefficient between $m$ and $\widetilde{r}_{1,t+1}$ whose value is in the interval $[-1,\, 1]$.

Eqs.(19) and (20) can be combined as:

$$\left| \frac{\mathrm{E}_0(\widetilde{r}_{1,t+1}) - r_{2,t+1}}{\sigma_0(\widetilde{r}_{1,t+1})} \right| \leq \frac{r_{2,t+1}\widehat{\alpha}\phi_t}{(1 - \widehat{\alpha}\phi_t)\sigma_0(\widetilde{r}_{1,t+1})} + \sigma_0(m)r_{2,t+1} \tag{21}$$

where we have used eq.(15) to approximate $\mathrm{E}_0(m)$ as $1/r_{2,t+1}$.

Adopt the constant relative risk aversion utility function $u(c) = c^{1-\gamma}/(1-\gamma)$. Thus $m = \beta(c_{t+1}/c_t)^{-\gamma}$. Now suppose that $\ln(c_{t+1}/c_t)$ is distributed normally with mean and standard deviation $\mathrm{E}_0[\ln(c_{t+1}/c_t)] = 0.02$ and $\sigma_0[\ln(c_{t+1}/c_t)] = 0.01$. From this it results that $\sigma_0(m) \approx 0.01\gamma$. Thus the upper limit of relation (21) is approximately:

$$\frac{\mathrm{E}_0(\widetilde{r}_{1,t+1}) - r_{2,t+1}}{\sigma_0(\widetilde{r}_{1,t+1})} \leq \frac{r_{2,t+1}\widehat{\alpha}\phi_t}{(1 - \widehat{\alpha}\phi_t)\sigma_0(\widetilde{r}_{1,t+1})} + 0.01\gamma r_{2,t+1} \tag{22}$$

Typical values are $\mathrm{E}_0(\widetilde{r}_{1,t+1}) = 1.07$, $\sigma_0(\widetilde{r}_{1,t+1}) = 0.15$ and $r_{2,t+1} = 1.01$. If $\widehat{\alpha} = 0$ then the risk aversion coefficient required in relation (22) is $\gamma = 39.6$. This value of $\gamma$ is much greater than usually estimated, which is the traditional equity premium puzzle (Cochrane, 2001, chap. 1). On the other hand, for $\phi_t = +1$, if $\widehat{\alpha} = 0.055$ then relation (22) implies $\gamma = 0.80$ which is an entirely ordinary level of Arrow-Pratt risk aversion.

The info-gap equity premium relations, eqs.(17), (18) and (22), each reveal a different aspect of the role of robust-satisficing in understanding the equity premium. Together they constitute the info-gap robust-satisficing version of the "empty fig. 4" in Mehra and Prescott's famous paper (1985). Mehra and Prescott's fig. showed that reasonable values of risk aversion and discount rate cannot be reconciled with observed equity premium, risk-free rate and consumption growth through a specific utility-maximizing paradigm. Eqs.(17), (18) and (22) demonstrate that these quantities are coherent within an info-gap robust-satisficing framework.

# 5 Stationarity

**Info-gap robust-satisficing stationarity** at time $t = 0$ is the existence of a set of prices $p$ and investments $x$ and a utility aspiration $\overline{U}$ such that $x$ is a robust-satisficing investment $\widehat{x}(\overline{U})$ which maximizes the robustness, eq.(6), and the budget-constrained consumptions are non-negative for all payoffs at horizons of uncertainty $\alpha \leq \widehat{\alpha}(x, \overline{U})$. In other words, at robust-satisficing stationarity the representative agent satisfices the discounted utility, maximizes the robustness to payoff-uncertainty, and has non-negative consumption within the budget if the realized uncertainty does not exceed the robustness.

The existence of robust-satisficing stationarity was assumed, eq.(11), in deriving the asset price relations, eqs.(12) and (13). In this section we identify conditions under which this assumption holds.

It is convenient to combine eqs.(5) and (8) to express the robustness of investment stream $x$, with utility aspiration $\overline{U}$, as:

$$\widehat{\alpha}(x, \overline{U}) = \max\left\{\alpha : \ \mu(\alpha, x, p) \geq \overline{U}\right\} \tag{23}$$

We continue with the assumptions that $u(c)$ is continuous and $u'(c) > 0$ and $\phi_{t-1}\widetilde{y}_{1t}x_{1,t-1} > 0$ for $t > 0$. Thus, from eq.(9), $\mu(\alpha, x, p)$ is continuous in both $x$ and $\alpha$ and strictly decreases as $\alpha$ increases. The robustness, $\widehat{\alpha}(x, \overline{U})$, of investments $x$ with aspiration $\overline{U}$ is the greatest value of $\alpha$ satisfying $\mu(\alpha, x, p) \geq \overline{U}$. Hence:

$$\mu(\alpha, x, p) = \overline{U} \quad \text{implies} \quad \widehat{\alpha}(x, \overline{U}) = \alpha \tag{24}$$

Finally, an investment stream $\widehat{x}$ which maximizes $\mu(\alpha, x, p)$ also maximizes $\widehat{\alpha}(x, \overline{U})$, where the value of the aspiration, $\overline{U}$, is $\mu(\alpha, \widehat{x}, p)$ and the corresponding maximal robustness is $\alpha$. We explain this as follows.

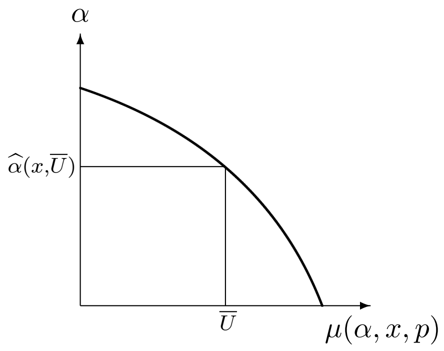

Figure 4: Robustness evaluated from $\mu(\alpha, x, p)$.

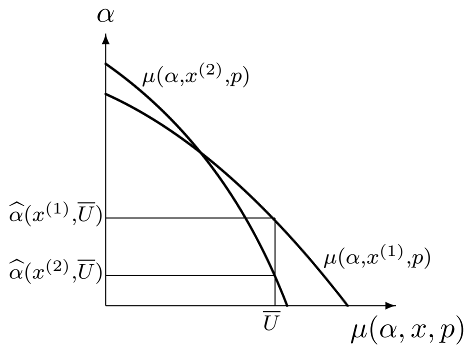

Figure 5: Illustration of eqs.(25) and (26).

Eq.(24) implies that a plot of $\alpha$ vs. $\mu(\alpha, x, p)$ is the same as a plot of $\widehat{\alpha}(x, \overline{U})$ vs. $\overline{U}$. This is illustrated in fig. 4 (which is the same a fig. 1).

Consider two different investment streams, $x^{(1)}$ and $x^{(2)}$, for which $\widetilde{c}_t(\alpha)$ in eq.(9) is non-negative for $t \geq 0$. From the monotonicity and continuity of $\mu(\alpha, x, p)$ in $\alpha$ we conclude that *if,* for some horizon of uncertainty $\alpha_1$:

$$\mu(\alpha_1, x^{(1)}, p) > \mu(\alpha_1, x^{(2)}, p) \tag{25}$$

*then:*

$$\widehat{\alpha}(x^{(1)}, \overline{U}) > \widehat{\alpha}(x^{(2)}, \overline{U}) = \alpha_1 \quad \text{for} \quad \overline{U} = \mu(\alpha_1, x^{(2)}, p) \tag{26}$$

This is illustrated in fig. 5.

Robust-satisficing investments, $\widehat{x}(\overline{U})$, maximize the robustness, eq.(6). From relations (25) and (26) we find that all robust-satisficing investments are those investment streams which maximize $\mu(\alpha, x, p)$ on $x$.

Finally we return to the concept of info-gap stationarity. From eq.(9) we recognize that $\mu(\alpha, x, p)$ is the $T$-period discounted utility with risky payoffs $(1-\alpha\phi_{t-1})\widetilde{y}_t$ and risk-free payoffs $y_{2t}$. Ljungqvist and Sargent (2000) discuss recursive methods for finding investment sequences $x$ which maximize the discounted utility. Since maximizing $\mu(\alpha, x, p)$ is equivalent to maximizing $\widehat{\alpha}(x, \overline{U})$, we see that the robustness has a maximum, info-gap robust-satisficing stationarity exists, and eq.(11) can be satisfied if and only if this discounted utility can be maximized. In other words, robust-satisficing stationarity exists if and only if there is an investment stream $\widehat{x}$ which maximizes this discounted utility. The utility aspiration at which the robustness is maximized is $\overline{U} = \mu(\alpha, \widehat{x}, p)$ and the maximal robustness is $\widehat{\alpha}(\widehat{x}, \overline{U}) = \alpha$.

# 6 Discussion

Info-gap robust-satisficing is a quantification of three concepts which have been discussed in economic literature: Knightian uncertainty, bounded rationality and robustness. Info-gap robust-satisficing explains the equity premium puzzle and suggests a method for modelling behavior under severe uncertainty.

**Knightian uncertainty.** An info-gap model of uncertainty is a non-probabilistic representation of uncertainty which is starkly minimalistic. Info-gap models provide no opportunity for insurance-like calculations. The theory of info-gap uncertainty provides one plausible quantitative model for Knight's concept of "true uncertainty" for which "there is no objective measure of the probability", as opposed to risk which is probabilistically measurable (Knight, 1921, pp.46, 120, 231–232). Info-gap models, together with the robust-satisficing decision model, have been used to explain the home-bias phenomenon (Ben-Haim and Jeske, 2003) and the Allais and Ellsberg paradoxes (Ben-Haim, 2001, sections 7.8 and 7.9). Info-gap models have also been employed in financial risk assessment (Ben-Haim, 2005a), modelling animal behavior under uncertainty (Carmel and Ben-Haim, 2005), project management (Ben-Haim and Laufer 1998), biological conservation (Regan *et al,* 2005), and engineering analysis and design (Ben-Haim, 2005).

Further discussion of the relation between Knight's conception and info-gap theory is found in (Ben-Haim, 2001, section 12.5). Similarly, Shackle's "non-distributional uncertainty variable" bears some similarity to info-gap analysis (Shackle, 1972, p.23). Likewise, Kyburg recognized the possibility of a "decision theory that is based on some non-probabilistic measure of uncertainty." (Kyburg, 1990, p.1094).

Info-gap models are not the only possible way to quantify Knightian uncertainty. On the contrary, Gilboa and Schmeidler (1989), Epstein and Wang (1994), Epstein and Miao (2003) and others, achieve uninsurable uncertainty of a clearly Knightian type by replacing a single prior probability distribution with a set of distributions. These approaches are Knightian "true uncertainty" since the absence of a probability measure on the set of probability distributions makes the uncertainty uninsurable. Nonetheless, an info-gap model of uncertainty is a more extreme departure from the probabilistic tradition. In our formulation, preferences are generated by the robustness function without any distribution functions at all.

**Bounded rationality and robustness.** The investor's ability to optimize future outcomes is severely limited. Past asset returns provide only a rough indication of future returns. The investor's access to information about and understanding of the relevant social and economic forces is far too limited to enable reliable assessment of maximal future behavior. Furthermore, the great variability of the returns motivates the choice of a strategy which balances aspiration for utility against aspiration for immunity to uncertainty. The info-gap robust-satisficing decision strategy does precisely that.

Hansen, Sargent and Tallarini (1999) "show how a preference for robustness lies concealed within the quantity implications of the permanent income model" (p.873). That is, a desire for robustness against uncertainty influences the choices of economic agents. This insight can be viewed as a starting point of the current paper, which shows that desire for robustness is as important as desire for material utility. In that spirit, the theorem which underlies the current paper establishes an irrevocable trade-off between robustness and reward. This trade-off motivates the robust-satisficing strategy, very deliberately violating the classical axiom of economic rationality. The investor does not attempt to maximize utility, or even a risk-adjusted utility. In the info-gap decision model, utility is satisficed; what is maximized is robustness to uncertainty. The "rationality" of this strategy is expressed in the trade-off of the aspiration for utility against the aspiration for robustness to uncertainty (illustrated in fig. 1). This trade-off is a general theorem which asserts that maximal utility is invariably accompanied by zero robustness. Utility-maximization is entirely unreliable.

**Equity premium.** Kocherlakota concludes his discussion of the equity premium puzzle with the comment:

> The *universality* of the equity premium tells us that, like money, the equity premium must emerge from some primitive and elementary features of asset exchange that are probably best captured through extremely stark models. With this in mind, we cannot hope to find a resolution to the equity premium puzzle by continuing in our current mode of patching the standard models of asset exchange with transactions costs here and risk aversion there. Instead, we must seek to identify what fundamental features of goods and asset markets lead to large risk adjusted price differences between stocks and bonds. (Kocherlakota, 1996, p.67)

The info-gap robust-satisficing decision model offers the possibility of new insight. Two ideas are prominent. First, risk aversion is a multi-faceted phenomenon which is not captured entirely by utility-function curvature. Especially in situations of great Knightian uncertainty, where betting is not a particularly useful concept, risk aversion is expressed in part by the willingness to forego utility in exchange for robustness-to-failure. Second, the info-gap robust-satisficing decision model demonstrates that the classical axiom of utility maximization can be relaxed without losing touch with economic intuition and data. The fundamental psychological premise — that utility is desirable — does not imply that maximal utility is most desirable. In this paper we have shown that realistic and sensible economic reasoning can be based on, and modelled by, the concept of satisficing the utility and maximizing the robustness with non-probabilistic info-gap models of uncertainty.

**Acknowledgements**

The author is indebted to critical comments by Michael Ben-Gad, Mark Burgman, Karsten Jeske, Doron Kliger, Jim Nason, Dan Peled, Will Roberds, Ellis Tallman, and Gady Zohar.

# 7 References

Ben-Haim, Yakov, 2001, *Information-gap Decision Theory: Decisions Under Severe Uncertainty,* Academic Press, San Diego.

Ben-Haim, Yakov, 2005, Info-gap Decision Theory For Engineering Design. Or: Why 'Good' is Preferable to 'Best', in E.Nikolaidis, D.Ghiocel and S.Singhal, eds., *Engineering Design Reliability Handbook,* CRC Press, Boca Raton.

Ben-Haim, Yakov, 2005a, Value at risk with Info-gap uncertainty, *Journal of Risk Finance,* to appear.

Ben-Haim, Yakov and Karsten Jeske, 2003, Home-bias in financial markets: Robust satisficing with info-gaps, Federal Reserve Bank of Atlanta, Working Paper Series, 2003-35, Dec. 2003. SSRN abstract and full paper at: http://ssrn.com/abstract=487585.

Ben-Haim, Yakov and Alexander Laufer, 1998, Robust reliability of projects with activity-duration uncertainty, *ASCE Journal of Construction Engineering and Management.* 124: 125–132.

Blanchard, Olivier Jean and Stanley Fischer, 1989, *Lectures on Macroeconomics,* MIT Press, Cambridge, MA.

Carmel, Yohay and Yakov Ben-Haim, 2005, Info-gap robust-satisficing model of foraging behavior: Do foragers optimize or satisfice?, *American Naturalist,* to appear.

Cochrane, John H., 2001, *Asset Pricing,* Princeton University Press, Princeton.

Epstein, Larry G. and Jianjun Miao, 2003, A two-person dynamic equilibrium under ambiguity, *Journal of Economic Dynamics and Control,* 27: 1253–1288.

Epstein, Larry G. and Tan Wang, 1994, Intertemporal asset pricing under Knightian uncertainty, *Econometrica,* 62: 283–322.

Gilboa, I. and D. Schmeidler, 1989, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics,* 18: 141–153.

Hansen, L. P., T. J. Sargent, and T. Tallarini (1999). Robust permanent income and pricing. *Review of Economic Studies,* 66, 873–907.

Knight, Frank H., 1921, *Risk, Uncertainty and Profit.* (Houghton Mifflin Co. Re-issued by University of Chicago Press, 1971).

Kyburg, Henry E. jr., 1990, Getting fancy with probability, *Synthese,* 90: 189–203.

Kocherlakota, Narayana R., The equity premium: It's still a puzzle, *Journal of Economic Literature,* 34: 42–71 (1996).

Ljungqvist, Lars and Thomas J. Sargent, 2000, *Recursive Macroeconomic Theory,* MIT Press, Cambridge, USA.

Mehra, Rajnish and Edward C.Prescott, 1985, The equity premium: A puzzle, *Journal of Monetary Economics,* 15: 145–161.

Regan, Helen M., Yakov Ben-Haim, Bill Langford, Will G. Wilson, Per Lundberg, Sandy J. Andelman, Mark A. Burgman, 2005, Robust decision making under severe uncertainty for conservation management, *Ecological Applications,* to appear.

Shackle, G.L.S., 1972, *Epistemics and Economics: A Critique of Economic Doctrines,* (Cambridge University Press, re-issued by Transaction Publishers, New Brunswick, 1992).