Info-gap Demand Theory: Economic Optimization Revisited

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Abstract

Neoclassical consumer demand theory is based on two basic optimization problems: utility maximization and expenditure minimization. What the theory lacks, however, is treatment of knightian "true" and "unmeasurable uncertainty" as distinct from insurable probabilistic risk. This paper presents two alternative optimization problems from which are derived two demand functions analogous in meaning to the neoclassical Walrasian and Hicksian demand functions. These alternative optimization problems include explicit treatment of non-probabilistic uncertainty and are based on the robust-satisficing and opportune-windfalling functions of information-gap decision theory. The solution of the robust satisficing optimization problem is a wealth-compensated demand function which obeys Walras' law and the law of price-demand trade-off.

Contents

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1 Introduction

Economics is the foremost social science of optimization. Neoclassical consumer demand theory is based on two basic optimization problems: utility maximization and expenditure minimization. The Walrasian and Hicksian demand functions and much of the associated theory result from these two optimization problems. What the theory lacks, however, is treatment of what Knight called "true" and "unmeasurable uncertainty" as distinct from measurable (and hence insurable) probabilistic risk. This paper presents two alternative optimization problems from which are derived two demand functions analogous in meaning to the Walrasian and Hicksian functions. These alternative optimization problems include explicit treatment of non-probabilistic uncertainty and are based on the robust-satisficing and opportune-windfalling functions of information-gap decision theory.

The attractiveness of the neo-classical economic theory of demand resides in the fact that meaningful theorems with observable manifestations are derived from two optimizations which rest upon fundamental suppositions about rational choice. Most prominently, these suppositions are that the consumer has sufficient knowledge to adopt preferences for all pairs of options, and that these preferences are sufficiently consistent to assure transitivity. Alternatively, the slightly less committal 'weak axiom of revealed preference' supposes that, if two options are accessible, then changes in price or wealth will not cause first one option and then the other to be selected. Economists have long recognized the importance of ill-defined information-gaps which lurk behind all quantitative analysis. (The term "information-gap" appears occasionally in economic literature.) As a prelude to our treatment of uncertainty, section 2 carefully examines some of the characterizations of unmeasurable uncertainties in economics.

In section 3 we briefly review the mathematics of information-gap models. An info-gap is the disparity between what *is known* and what *could be known.* This concept is useful in representing the imperfect knowledge and bounded rationality of micro-economic agents. No measure functions are involved, and the sparseness of info-gap models matches the paucity of information common to many decision situations. In section 4 we outline consumer-choice optimization problems, both in the classical framework and in the context of info-gap decision theory. Info-gap theory avoids the standard assumptions of complete, transitive preferences. This is pertinent to the attempt to build a theory which corresponds to knowledge-deficient decision making. Our main results appear in sections 5 and 6, in which we establish the info-gap versions of the Hicksian and Walrasian demand functions. Proposition 1 in section 5 establishes the info-gap version of the compensated law of demand. Propositions 2 and 3 in section 6 lay the theoretical basis for identifying wealth-compensation, as well as other implications of this info-gap economic analysis. These propositions show that the phenomenology of classical demand theory is retained in a theory which treats imperfect knowledge in a distinctive manner relevant to much micro-economic behavior. Section 7 contains several examples. All proofs appear in the appendix.

2 Uncertainty in Economics

There has been enormous progress in the economics of deficient information during the past half century. However, this progress has been based almost exclusively upon probabilistic models of uncertainty. Long before these developments, Knight distinguished very explicitly between 'true', 'unmeasurable' uncertainty, and probabilistic risk:

> Uncertainty must be taken in a sense radically distinct from the familiar notion of Risk, from which it has never been properly separated. The term 'risk', as loosely used in

> everyday speech and in economic discussion, really covers two things which, functionally at least, in their causal relations to the phenomena of economic organization, are categorically different. ... It will appear [in later chapters] that a *measurable* uncertainty, or 'risk' proper, as we shall use the term, is so far different from an *unmeasuable* one that it is not in effect an uncertainty at all. We shall accordingly restrict the term 'uncertainty' to cases of the non-quantitative type. It is the 'true' uncertainty, and not risk, as has been argued, which forms the basis of a valid theory of profit and accounts for the divergence between actual and theoretical competition." [1, pp.19–20, italics in the original]

In practical terms, Knight repeatedly argued that the uncertainties upon which entrepreneurial competition thrives are utterly different from probabilities:

> The uncertainties which persist as causes of profit are those which are uninsurable because there is no objective measure of the probability of gain or loss. ... Situations in regard to which business judgment must be exercised do not repeat themselves with sufficient conformity to type to make possible a computation of probability. [2, p.120]

The distinction between probabilistic risk and generic uncertainty has been widely accepted by economists. Samuelson notes that:

> Thinkers have naturally questioned whether the phenomena of *uncertainty* can be usefully handled by the quasi-mathematical notions of 'probability.' Certain subsets of uncertainty — those dealing with risks, gambling, insurance, repetitive inventory and quality control of production, and even with tactics of repeated investing — are thought to lend themselves better to useful employment of probability procedures. [3, pp.503–504, italics in the original]

Nobody disputes that probability distributions reflect imperfect knowledge. The point is that real economic uncertainties, those which motivate the entrepreneur and are either a blessing or a bane, are often starker and sparser than is reflected in frequentist or Bayesian/subjectivist measure functions. Real economic uncertainty "is the complement of knowledge. It is the gap between what is known and what needs to be known to make correct decisions." [4, p.1]. Uncertainty is an information gap: "the difference between the amount of information required to perform the task and the amount of information already possessed by the organization." [5, p.5].

Shackle explains that info-gaps arise as a necessary epistemic consequence of intelligent learning:

> This insufficiency of knowledge is permanent and part of the nature of things, for consciousness consists precisely in the continuous gaining of knowledge. [6, pp.3–4]

Consequently, Shackle continues, the enterpriser's

> duty is to fill, with inventions and figments, the gap between what can be known and what needs to be known. When there is no such gap, there need be no enterpriser in the sense of policy-originator. [6, p.145]

Keynes stated the same idea somewhat differently:

> The outstanding fact is the extreme precariousness of the basis of knowledge on which our estimates of prospective yield have to be made. Our knowledge of the factors which will govern the yield of an investment some years hence is usually very slight and often negligible. [7, p.149]
>
> Nor can we rationalise our behaviour by arguing that to a man in a state of ignorance errors in either direction are equally probable, so that there remains a mean actuarial

> expectation based on equi-probabilities. For it can easily be shown that the assumption of arithmetically equal probabilities based on a state of ignorance leads to absurdities. [7, p.152]

Hayek has also made this point, from the rather different perspective of modelling and optimizing the naturally regulated behavior which is characteristic of competitive markets:

> What is the problem we wish to solve when we try to construct a rational economic order? On certain familiar assumptions the answer is simple enough. *If* we possess all the relevant information, *if* we command complete knowledge of available means, the problem which remains is purely one of logic. ...
>
> This, however, is emphatically *not* the economic problem which society faces. ...
>
> The peculiar character of the problem of a rational economic order is determined precisely by the fact that the knowledge of the circumstances of which we must make use never exists in concentrated or integrated form but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess. [8, p.77, italics in the original]

Simon has stated much the same idea, expressing it against the backdrop of modern utility theory:

> Global rationality, the rationality of neoclassical theory, assumes that the decision maker has a comprehensive, consistent utility function, knows all the alternatives that are available for choice, can compute the expected value of utility associated with each alternative, and chooses the alternative that maximizes expected utility. Bounded rationality, a rationality that is consistent with our knowledge of actual human choice behavior, assumes that the decision maker must search for alternatives, has egregiously incomplete and inaccurate knowledge about the consequences of actions, and chooses actions that are expected to be satisfactory (attain targets while satisfying constraints). [9, p.17]

March has presented the same criticism of traditional decision theory in the context of organizational decision-making [10, p.12]:

> Within decision theory, preferences are treated as important but unproblematic. A decision-maker is assumed to have preferences that are consistent, stable, and exogenous to the choice process. Observations of organizations suggest that preferences are often far from consistent, stable, or exogenous.

Economists have long been aware of the critique of probability provided by the workers cited here as well as by others. Since virtually all treatments of economic uncertainty have been probabilistic, some writers have found it necessary to provide an explanation for why these treatments can be expected to work at all. Mas-Colell, Whinston and Green suggest that the Bayesian approach, based on personal subjective probabilities, has usually underwritten these attempts "by reducing all uncertainty to risk through the use of beliefs expressible as probabilities." [11, p.207]. This, however, does not answer the critique of probability, it only sidesteps it by assuming that there is actually no distinction at all between uncertainty and risk. Arrow follows Knight in accepting the reality of the distinction, and provides a revealing explanation for why uncertainty has not been explicitly included in economic analysis. Arrow accepts in a "fundamental sense" that the "seemingly mechanical nature of the probability calculus ... [leads to its] failure to reflect the tentative, creative nature of the human mind in the face of the unknown." Arrow suggests, however, that this "seems to lead only to the conclusion that no theory can be formulated for this case." [12, p.19].

3 Info-gap Models of Uncertainty

Our quantification of knowledge-deficiency is based on non-probabilistic information-gap models [13]. An info-gap is a disparity between what the decision maker knows and what could be known. The range of possibilities expands as the info-gap grows. An info-gap model is a family of nested sets. Each set corresponds to a particular degree of knowledge-deficiency, according to its level of nesting. Each element in a set represents a possible event. There are no measure functions in an info-gap model.

Info-gap theory provides a quantitative model for Knight's concept of "true uncertainty" for which "there is no objective measure of the probability", as opposed to risk which is probabilistically measurable [1, pp.46, 120, 231–232]. Further discussion of the relation between Knight's conception and info-gap theory is found in [13, section 12.5]. Similarly, Shackle's "non-distributional uncertainty variable" bears some similarity to info-gap analysis [6, p.23]. Likewise, Kyburg recognized the possibility of a "decision theory that is based on some non-probabilistic measure of uncertainty." [14, p.1094].

Events are represented as vectors or vector functions f. Knowledge-deficiency is expressed at two levels by info-gap models. For fixed α the set $\mathcal{F}(\alpha, \widetilde{f})$ represents a degree of variability of f around the centerpoint $\widetilde{f}$. A greater value of α entails a greater range of possible variation of f, so α is called the *uncertainty parameter* and expresses the information gap between what is known ($\widetilde{f}$ and the structure of the sets) and what needs to be known for an ideal solution (the exact value of f). The value of α is usually unknown, which constitutes the second level of imperfection of knowledge: the horizon of variation is unbounded.

Let $\Re_+$ denote the non-negative real numbers and let Ω be a Banach space in which the uncertain quantities f are defined. An info-gap model $\mathcal{F}(\alpha, \widetilde{f})$ is a map from $\Re_+ \times \Omega$ into the power set of Ω. Info-gap models obey two basic axioms. *Nesting:* $\mathcal{F}(\alpha, \widetilde{f}) \subset \mathcal{F}(\alpha', \widetilde{f})$ if $\alpha < \alpha'$. *Contraction:* $\mathcal{F}(0, \widetilde{f})$ is the singleton set $\{\widetilde{f}\}$. Nesting is the most characteristic of the info-gap axioms. It expresses the intuition that possibilities expand as the info-gap grows. For more discussion of these axioms see [15].

4 Consumer-Choice Optimization Problems

In section 4.1 we briefly review the utility maximization and expenditure minimization problems from which neoclassical demand theory arises, and discuss the intuitions which underlie these strategies of choice. Section 4.2 is devoted to a discussion of the info-gap robustness and opportunity functions. We employ these decision functions in section 4.3 to formulate two info-gap optimization problems — robust satisficing and opportune windfalling — and explain their analogy to the traditional choice problems.

Notation: if $x \in \Re^L$, then $x > 0$ means $x_i > 0$ for all $i = 1, \ldots, L$. Also, $x \geq 0$ means $x_i \geq 0$ for all $i = 1, \ldots, L$.

4.1 Neoclassical Consumer-Choice Optimization Problems

Our discussion follows [16]. A preference relation $\succeq$ on $X \subset \Re^L$ is **rational** if the following properties hold:

(i) **Completeness:** For all x, $y \in X$, $x \succeq y$ or $y \succeq x$ or both.

(ii) **Transitivity:** For all x, y, $z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

A scalar function $u(x)$ is a **utility function** representing the preference relation $\succeq$ if, for all $x, y \in X$, $x \succeq y$ if and only if $u(x) \geq u(y)$.

Given prices $p > 0$ and utility $u > u(0)$, the **expenditure minimization problem** (EMP) is:

$$\min_{x \geq 0} p^T x \quad \text{subject to} \quad u(x) \geq u \tag{1}$$

Given prices $p > 0$ and wealth $w > 0$, the **utility maximization problem** (UMP) is:

$$\max_{x \geq 0} u(x) \quad \text{subject to} \quad p^T x \leq w \tag{2}$$

A vector x which solves the EMP is a Hicksian demand function, $h(p, u)$, while a solution of the UMP is a Walrasian demand function, $x(p, w)$.

A central result in demand theory states that, if $u(x)$ is a continuous utility function representing a locally non-satiated preference relation, then if x^* is an optimal choice for the UMP with wealth $w > 0$, then it is optimal for the EMP when the required utility is $u(x^*)$. Likewise, if x^* is an optimal choice for the EMP when the required utility is $u > u(0)$, then it is optimal for the UMP with wealth $w = p^T x^*$ [16, Proposition 3.E.1, p.58]. In otherwords, these two choice problems coalesce for appropriate choices of the parameters.

Nonetheless, different intuitions motivate these two selections of a consumption vector x. This is important for our study since it is the intuitions which we wish to preserve in the info-gap context.

The UMP is an ambitious program of maximizing satisfaction subject to a budget constraint. The UMP "squeezes the orange" as much as possible. And in fact any solution x^* satisfies Walras' law, $p^T x^* = w$, meaning that the budget is exploited to the limit [16, Proposition 3.D.2, pp.51–52].

The EMP, on the other hand, is motivated by (possibly cautious) satisficing. The decision maker chooses a (possibly modest) level of utility u and pinches the budget as much as possible to just achieve this level of utility. And in fact any solution x^* does exactly that: $u(x^*) = u$; there is no excess utility [16, Proposition 3.E.3, p.61].

4.2 Robustness and Opportunity

We now present two info-gap optimization problems and explain their conceptual proximity to the intuitions underlying the neoclassical EMP and UMP.

The decision maker will choose a commodity vector $x \in X \subseteq \Re^L$. The outcome of this choice is influenced by an unknown vector (or vector function) $f \in \Omega$ whose range of possible variation is represented by an info-gap model $\mathcal{F}(\alpha, \widetilde{f})$, $\alpha \geq 0$, in a Banach space Ω.

We will define two real-valued reward functions defined on (x, A) where $x \in X$ and $A \subset \Omega$. The **lower reward function** $\mathcal{R}_*(x, A)$ is monotonically decreasing on the power set of Ω, at fixed choice x:

$$A \subset B \quad \text{implies} \quad \mathcal{R}_*(x, A) \geq \mathcal{R}_*(x, B) \tag{3}$$

The **upper reward function** $\mathcal{R}^*(x, A)$ is monotonically increasing on the power set of Ω, at fixed choice x:

$$A \subset B \quad \text{implies} \quad \mathcal{R}^*(x, A) \leq \mathcal{R}^*(x, B) \tag{4}$$

The monotonicity of these reward functions does not relate to the commodity vector x, or to the decision maker's preferences regarding these choices. The reward functions are monotonic in the space Ω of unknown auxiliary events f, and expresses the impact of this ambient variation on the available outcomes of a choice. Properties (3) and (4) do not establish preference relations on x.

The most common realizations of the lower and upper reward functions is in terms of least and greatest available rewards as f varies within a set $A \subset \Omega$. Let $r(x, f)$ be the reward actually realized when the choice is $x \in X$ and the unknown quantity takes the value $f \in \Omega$. Common choices of lower and upper reward functions are:

$$\mathcal{R}_*(x, A) = \min_{f \in A} r(x, f) \quad (5)$$

$$\mathcal{R}^*(x, A) = \max_{f \in A} r(x, f) \quad (6)$$

Thus $\mathcal{R}_*(x, A)$ is the least available reward, while $\mathcal{R}^*(x, A)$ is the greatest available reward, in the uncertain environment A. In this realization, $\mathcal{R}_*$ and $\mathcal{R}^*$ still have not determined a preference relation on x, both because the set A is undetermined and because the lower and upper rewards may behave differently.

More specifically, we will subsequently choose the set A as a set $\mathcal{F}(\alpha, \widetilde{f})$ in an info-gap family of nested sets. Then, in eqs.(5) and (6), $\mathcal{R}_*[x, \mathcal{F}(\alpha, \widetilde{f})]$ is the least accessible reward up to info-gap α, while $\mathcal{R}^*[x, \mathcal{F}(\alpha, \widetilde{f})]$ is the greatest accessible reward up to α.

Like the neoclassical utility function $u(x)$, the reward functions $\mathcal{R}_*(x, A)$ and $\mathcal{R}^*(x, A)$ are real-valued and represent desirable reward. The decision maker prefers more rather than less. Unlike $u(x)$, however, neither $\mathcal{R}_*(x, A)$ nor $\mathcal{R}^*(x, A)$ need be continuous or convex, (though we will sometimes assume continuity), nor do they derive from preference relations. Moreover, the decision maker can't know the values of $\mathcal{R}_*(x, A)$ and $\mathcal{R}^*(x, A)$ which will be realized in practice because the set A is unknown. Finally, while $\mathcal{R}_*$ and $\mathcal{R}^*$ entail knowledge-deficiency, they are not probabilistic but instead depend on an info-gap model.

We will now use $\mathcal{R}_*(x, A)$ and $\mathcal{R}^*(x, A)$ to define two decision functions [13] which generate preferences on values of x. These preferences are not unique, nor are the same preferences necessarily derived from each of the two decision functions. These preferences will vary with aspiration level, price, wealth and possibly other exogenous factors. These two decision functions are motivated by intuitions which are similar to those underlying the classical decision problems, EMP and UMP.

Let $r_{\rm c}$ be a value of reward which the decision maker strives to achieve; more reward would be better, but less than $r_{\rm c}$ would be unacceptable. The decision maker wishes to satisfice at reward level $r_{\rm c}$. The decision maker might choose or contemplate $r_{\rm c}$-values in the same way that minimal utility values u are chosen in the EMP of eq.(1). The **robustness** of choice x is the greatest level of knowledge-deficiency at which reward no less than $r_{\rm c}$ is guaranteed:

$$\widehat{\alpha}(x, r_{\rm c}) = \max \left\{ \alpha : \ \mathcal{R}_*[x, \mathcal{F}(\alpha, \widetilde{f})] \geq r_{\rm c} \right\} \quad (7)$$

$\widehat{\alpha}(x, r_{\rm c})$ is a robust satisficing decision function.

Let $r_{\rm w}$ be a large value of reward (much greater than $r_{\rm c}$) which the decision maker would be delighted to achieve; lower reward would be acceptable, but reward as large as $r_{\rm w}$ is a windfall success. The decision maker might evaluate or contemplate $r_{\rm w}$-values much as values of the maximum utility obtained in the UMP of eq.(2) are evaluated. The **opportunity** inherent in choice x is the least level of knowledge-deficiency at which windfall can occur:

$$\widehat{\beta}(x, r_{\rm w}) = \min \left\{ \alpha : \ \mathcal{R}^*[x, \mathcal{F}(\alpha, \widetilde{f})] \geq r_{\rm w} \right\} \quad (8)$$

$\widehat{\beta}(x, r_{\rm w})$ is an opportune windfalling decision function.

$\widehat{\alpha}(x, r_{\rm c})$ and $\widehat{\beta}(x, r_{\rm w})$ are **immunity functions**. $\widehat{\alpha}(x, r_{\rm c})$ is the immunity against failure (reward less than $r_{\rm c}$). When $\widehat{\alpha}(x, r_{\rm c})$ is large, failure can occur only at great ambient uncertainty; the decision maker is not vulnerable to pernicious uncertainty. Similarly, $\widehat{\beta}(x, r_{\rm w})$ is the immunity against windfall (reward no less than $r_{\rm w}$). When $\widehat{\beta}(x, r_{\rm w})$ is small, windfall can occur even under mundane circumstances; the decision maker is not immune to propitious uncertainty.

These considerations lead to preference rankings on the choice vector x. While "bigger is better" for robustness $\widehat{\alpha}(x, r_{\rm c})$, "big is bad" for opportunity $\widehat{\beta}(x, r_{\rm w})$. The preferences induced by the robust-satisficing strategy are:

$$x \succeq_{\mathrm{r}} x' \quad \text{if} \quad \widehat{\alpha}(x, r_{\mathrm{c}}) \geq \widehat{\alpha}(x', r_{\mathrm{c}}) \tag{9}$$

Likewise, the preferences induced by the opportune-windfalling strategy are:

$$x \succeq_{\mathrm{o}} x' \quad \text{if} \quad \widehat{\beta}(x, r_{\mathrm{w}}) \leq \widehat{\beta}(x', r_{\mathrm{w}}) \tag{10}$$

We note that neither $\succeq_{\mathrm{r}}$ nor $\succeq_{\mathrm{o}}$ is necessarily single-valued for any pair of choices x and x': the preferences may change with r_{c} and r_{w}, respectively. Moreover, $\succeq_{\mathrm{r}}$ and $\succeq_{\mathrm{o}}$ may rank the same pair of options differently. The preference relations $\succeq_{\mathrm{r}}$ and $\succeq_{\mathrm{o}}$, either alone or together, do not satisfy the rationality conditions of completeness and transitivity. They do not establish unique preferences for all pairs of available choices, and they hence do not entail transitivity of preference.

4.3 Info-gap Consumer-Choice Optimization Problems

We now formulate two info-gap optimization problems and explain their conceptual similarity to the neoclassical EMP and UMP.

Given prices $p > 0$, critical reward r_{c}, and demanded robustness $\widehat{\alpha}_{\mathrm{d}}$, the **robust satisficing problem** (RSP) is:

$$\min_{x \in X} p^T x \quad \text{subject to} \quad \widehat{\alpha}(x, r_{\mathrm{c}}) \geq \widehat{\alpha}_{\mathrm{d}} \tag{11}$$

Given prices $p > 0$, windfall reward r_{w}, and wealth $w > 0$, the **opportune windfalling problem** (OWP) is:

$$\min_{x \in X} \widehat{\beta}(x, r_{\mathrm{w}}) \quad \text{subject to} \quad p^T x \leq w \tag{12}$$

We now explain the intuitive similarity between robust satisficing (RSP) and expenditure minimization (EMP), and between opportune windfalling (OWP) and utility maximization (UMP).

RSP and EMP: The structural parallel between the EMP in eq.(1) and the RSP is obvious. In both, the expenditure $p^T x$ is minimized subject to a satisficing constraint on a preference-generating function: $u(x)$ or $\widehat{\alpha}(x, r_{\mathrm{c}})$. Furthermore, the RSP also satisfices the reward by means of the inequality on the lower reward function $\mathcal{R}_*$ in eq.(7). The critical value of utility u in the EMP or the critical reward r_{c} and demanded robustness $\widehat{\alpha}_{\mathrm{d}}$ in the RSP can be chosen small or large at the decision maker's discretion. In both cases the decision maker is adopting a cautious or protective stance while attempting to guarantee a specified minimal level of satisfaction.

The fundamental difference between the RSP and the EMP is that the latter is based on a complete, transitive preference relation while the former is not. The EMP presumes substantially greater knowledge by the decision maker about the options and their import. The robustness function $\widehat{\alpha}(x, r_{\mathrm{c}})$ generates a preference relation contingent upon a choice of the critical reward, r_{c}. Preferences may change as the decision maker's aspiration for reward, r_{c}, changes. The robustness function is not directly a utility, but an auxiliary evaluation of feasibility or justifiability of option x with regard to reward-aspiration r_{c}. Knowing a consumer's robustness function tells us less about the utility to that individual of option x, than knowing a utility function $u(x)$.

OWP and UMP: The UMP in eq.(2) is structurally and conceptually similar to the OWP once we recognize that a minimum $\widehat{\beta}$ optimizes the opportunity for sweeping windfall success. This 'windfalling' is the info-gap analog of the classical search for maximum utility. Moreover, both $\max u(x)$ and $\min \widehat{\beta}$ are subject to the same budget constraint, $p^T x \leq w$. Furthermore, we explained in discussing eq.(2) that the UMP is an ambitious strategy, attempting to achieve the greatest utility facilitated by the available budget. There is no margin of safety in the UMP; large utility is its own protection. Similarly, the windfalling strategy aspires to facilitate reward as large as r_w, expressed by the inequality on the upper reward function $\mathcal{R}^*$ in eq.(8). In the info-gap context there may be no maximal reward even given a finite budget because of the endless potential for propitious uncertainty. What the windfalling decision maker is doing is optimizing the possibility of exploiting this potential, while attempting to facilitate great reward r_w far in excess of the critical reward r_c needed for survival.

The primary distinction between the OWP and the UMP, as in the RSP and EMP case, is that the opportunity function $\widehat{\beta}(x, r_w)$ is not itself a utility function, but rather an auxiliary assessment of the propitiousness of option x regarding windfall aspiration r_w.

Viewing the four optimization problems together, the primary difference is the symmetry in EMP/UMP which is broken in RSP/OWP by $\widehat{\alpha}(x, r_c)$ and $\widehat{\beta}(x, r_w)$. Implications of this, together with some substantive differences between robustness and opportunity, will arise in propositions 2 and 3.

5 Price-Demand Antagonism

In demand theory, $h(p, u)$ is a Hicksian demand function if it is a solution of the EMP. That is, $x = h(p, u)$ minimizes $p^T x$ subject to $u(x) \geq u$. If $u(x)$ is a continuous utility function representing a locally non-satiated preference relation, then $h(p, u)$ obeys the compensated demand law [16, Proposition 3.E.4, p.62]:

$$(p - p')^T [h(p, u) - h(p', u)] \leq 0 \tag{13}$$

Let us define $\widehat{x}_r(p, \widehat{\alpha}_d, r_c)$ as a consumer choice x which solves the RSP. That is, $p^T \widehat{x}_r(p, \widehat{\alpha}_d, r_c)$ is minimal subject to the robustness constraint $\widehat{\alpha}(x, r_c) \geq \widehat{\alpha}_d$. $\widehat{x}_r(p, \widehat{\alpha}_d, r_c)$ is the **robust satisficing demand.** In the theory we are developing here, $\widehat{x}_r(p, \widehat{\alpha}_d, r_c)$ is the analog of $h(p, u)$. The analog of the classical Hicksian demand law is:

Proposition 1 Given: *$\mathcal{F}(\alpha, \widetilde{f})$ is an info-gap model in the Banach space Ω; $\widehat{\alpha}(x, r_c)$ is a robustness function for $\mathcal{F}(\alpha, \widetilde{f})$ based on a lower reward function $\mathcal{R}_*(x, A)$ which is monotonically decreasing on the sets in Ω; $\widehat{x}_r(p, \widehat{\alpha}_d, r_c)$ is a robust satisficing demand function for $\widehat{\alpha}(x, r_c)$.*

Then: *for any price vectors p and p' we have:*

$$(p - p')^T [\widehat{x}_r(p, \widehat{\alpha}_d, r_c) - \widehat{x}_r(p', \widehat{\alpha}_d, r_c)] \leq 0 \tag{14}$$

That is, like the Hicksian demand $h(p, u)$, price p and robust satisficing demand $\widehat{x}_r$ vary antagonistically. Moreover, subsequent results will demonstrate that, like $h(p, u)$, the robust satisficing demand $\widehat{x}_r(p, \widehat{\alpha}_d, r_c)$ is wealth-compensated.

Let us define $\widehat{x}_o(p, w, r_w)$ as a consumer choice x which solves the OWP. That is, $\widehat{\beta}(\widehat{x}_o, r_w)$ is a minimal value of the opportunity function subject to the budget constraint $p^T \widehat{x}_o \leq w$. Since the OWP is the info-gap analog of the UMP, we see that $\widehat{x}_o(p, w, r_w)$ is the analog of the Walrasian demand function.

6 Wealth-Compensated Demand

Definition 1 *The function robustness $\widehat{\alpha}(x, r)$ is* **continuous** *at x if, for each $\varepsilon > 0$ there is a $\delta > 0$ such that:*

$$|\widehat{\alpha}(x, r) - \widehat{\alpha}(x', r)| < \varepsilon \quad \text{whenever} \quad \|x - x'\| < \delta \tag{15}$$

Continuity of the opportunity function is similarly defined.

Relations (3) and (4) define the lower and upper reward functions as monotonic in the uncertainty sets. The following property of strict monotonicity is different.

Definition 2 *The lower reward function* $\mathcal{R}_*(x, A)$ *is* **strictly monotonic** *in* α *for info-gap model* $\mathcal{F}(\alpha, \widetilde{f})$ *if, for all* $x \in X$ *and all* $\widetilde{f} \in \Omega$*:*

$$\alpha < \alpha' \quad \text{implies} \quad \mathcal{R}_*[x, \mathcal{F}(\alpha, \widetilde{f})] > \mathcal{R}_*[x, \mathcal{F}(\alpha', \widetilde{f})] \tag{16}$$

Definition 3 *The upper reward function* $\mathcal{R}^*(x, A)$ *is* **strictly monotonic** *in* α *for info-gap model* $\mathcal{F}(\alpha, \widetilde{f})$ *if, for all* $x \in X$ *and all* $\widetilde{f} \in \Omega$*:*

$$\alpha < \alpha' \quad \text{implies} \quad \mathcal{R}^*[x, \mathcal{F}(\alpha, \widetilde{f})] < \mathcal{R}^*[x, \mathcal{F}(\alpha', \widetilde{f})] \tag{17}$$

Definition 4 *The lower reward function* $\mathcal{R}_*(x, A)$ *is* **continuous** *in* α *at* $\widetilde{f}$ *for an info-gap model* $\mathcal{F}(\alpha, \widetilde{f})$ *if, for each* $x \in X$ *and for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that:*

$$\left| \mathcal{R}_*(x, \mathcal{F}(\alpha, \widetilde{f})] - \mathcal{R}_*(x, \mathcal{F}(\alpha', \widetilde{f})] \right| < \varepsilon \quad \text{whenever} \quad |\alpha - \alpha'| < \delta \tag{18}$$

Continuity of an upper reward function is similarly defined.

Definition 5 *Lower and upper reward functions* $\mathcal{R}_*(x, A)$ *and* $\mathcal{R}^*(x, A)$ *are* **similarly ordered** *if, for all* $x, x' \in X$ *and for all* $A \subset \Omega$*:*

$$\mathcal{R}_*(x, A) < \mathcal{R}_*(x', A) \quad \text{if and only if} \quad \mathcal{R}^*(x, A) < \mathcal{R}^*(x', A) \tag{19}$$

An immunity function, $\widehat{\alpha}(x, r_{\rm c})$ or $\widehat{\beta}(x, r_{\rm w})$, is 'non-satiated' if an arbitrarily small change in x can improve the immunity. Since "bigger is better" for robustness $\widehat{\alpha}(x, r_{\rm c})$, while "big is bad" for opportunity $\widehat{\beta}(x, r_{\rm w})$, the definitions of non-satiation for these immunity functions are different but symmetrical.

Definition 6 *The opportunity function is* **non-satiated** *at* $x \in X$ *if, for every* $\varepsilon > 0$*, there is an* $x' \in X$ *such that*

$$\|x - x'\| < \varepsilon \quad \text{and} \quad \widehat{\beta}(x', r_{\rm w}) < \widehat{\beta}(x, r_{\rm w}) \tag{20}$$

Definition 7 *The robustness function is* **non-satiated** *at* $x \in X$ *if, for every* $\varepsilon > 0$*, there is an* $x' \in X$ *such that*

$$\|x - x'\| < \varepsilon \quad \text{and} \quad \widehat{\alpha}(x', r_{\rm w}) > \widehat{\alpha}(x, r_{\rm w}) \tag{21}$$

Lemma 1 Given: $\mathcal{R}_*(x, A)$ *is a lower reward function which is continuous in* α *at all* $\widetilde{f}$ *for an info-gap model* $\mathcal{F}(\alpha, \widetilde{f})$*;* $\widehat{\alpha}(x, r_{\rm c})$ *is the robustness function defined for this reward function;* $\widehat{\alpha}(x, r_{\rm c})$ *is continuous at all* x*.*

If x *solves the RSP of eq.(11)* **then:**

$$\widehat{\alpha}(x, r_{\rm c}) = \widehat{\alpha}_{\rm d} \quad \text{and} \quad \mathcal{R}_*[x, \mathcal{F}(\widehat{\alpha}(x, r_{\rm c}), \widetilde{f})] = r_{\rm c} \tag{22}$$

The RSP entails satisficing both the robustness (the inequality in eq.(11)) and the reward (the inequality in eq.(7)). Lemma 1 asserts that there is no excess robustness or reward at a solution of the RSP.

Lemma 2 Given: $\mathcal{R}^*(x, A)$ *is an upper reward function which is continuous in* α *at all* $\widetilde{f}$ *for an info-gap model* $\mathcal{F}(\alpha, \widetilde{f})$*;* $\widehat{\beta}(x, r_{\rm w})$ *is the opportunity function defined for this reward function;* $\widehat{\beta}(x, r_{\rm w})$ *is continuous at all* x*.*

If x *solves the OWP of eq.(12)* **and if** $\widehat{\beta}(x, r_{\rm w})$ *is non-satiated at this* x*,* **then:**

$$p^T x = w \quad \text{and} \quad \mathcal{R}^*[x, \mathcal{F}(\widehat{\beta}(x, r_{\rm w}), \widetilde{f})] = r_{\rm w} \tag{23}$$

Note that the opportunity function is satiated if $\widehat{\beta}(x, r_{\rm w}) = 0$. Hence (23) need not hold if the minimal opportunity function vanishes.

Lemma 2 states that if x solves the OWP then x also obeys Walras' law of complete utilization of wealth. Furthermore, the lemma asserts that there is no excess reward in the solution of the OWP: the inequality in the definition of the opportunity function, eq.(8), is an equality at x.

Proposition 2 Given: *$\mathcal{R}_*(x, A)$ and $\mathcal{R}^*(x, A)$ are similarly ordered lower and upper reward functions, respectively; $\mathcal{R}_*(x, A)$ is continuous in α at all $\widetilde{f}$ for an info-gap model $\mathcal{F}(\alpha, \widetilde{f})$; $\mathcal{R}^*(x, A)$ is strictly monotonic in α; $\widehat{\alpha}(x, r_{\rm c})$ and $\widehat{\beta}(x, r_{\rm w})$ are robustness and opportunity functions, respectively, defined for these reward functions; $\widehat{\alpha}(x, r_{\rm c})$ and $\widehat{\beta}(x, r_{\rm w})$ are continuous at all x.*

If *$\widehat{x}_{\rm r}$ solves the RSP of eq.(11),* **then** *$\widehat{x}_{\rm r}$ solves the OWP of eq.(12) with $w = p^T \widehat{x}_{\rm r}$ and $r_{\rm w} = \mathcal{R}^*[\widehat{x}_{\rm r}, \mathcal{F}(\widehat{\alpha}_{\rm d}, \widetilde{f})]$. Moreover, $\widehat{\beta}(\widehat{x}_{\rm r}, r_{\rm w}) = \widehat{\alpha}_{\rm d} = \widehat{\alpha}(\widehat{x}_{\rm r}, r_{\rm c})$.*

Proposition 3 Given: *$\mathcal{R}_*(x, A)$ and $\mathcal{R}^*(x, A)$ are similarly ordered lower and upper reward functions, respectively; $\mathcal{R}^*(x, A)$ is continuous in α at all $\widetilde{f}$ for an info-gap model $\mathcal{F}(\alpha, \widetilde{f})$; $\mathcal{R}_*(x, A)$ is strictly monotonic in α; $\widehat{\alpha}(x, r_{\rm c})$ and $\widehat{\beta}(x, r_{\rm w})$ are robustness and opportunity functions, respectively, defined for these reward functions; $\widehat{\alpha}(x, r_{\rm c})$ and $\widehat{\beta}(x, r_{\rm w})$ are continuous at all x and $\widehat{\alpha}(x, r_{\rm c})$ is non-satiated at all x.*

If *$\widehat{x}_{\rm o}$ solves the OWP of eq.(12)* **and if** *$\widehat{\beta}(\widehat{x}_{\rm o}, r_{\rm w})$ is non-satiated at $\widehat{x}_{\rm o}$,* **then** *$\widehat{x}_{\rm o}$ solves the RSP of eq.(11) with $\widehat{\alpha}_{\rm d} = \widehat{\beta}(\widehat{x}_{\rm o}, r_{\rm w})$ and $r_{\rm c} = \mathcal{R}_*[\widehat{x}_{\rm o}, \mathcal{F}(\widehat{\beta}(\widehat{x}_{\rm o}, r_{\rm w}), \widetilde{f})]$. Moreover, $p^T \widehat{x}_{\rm o} = w$ and $\widehat{\alpha}(\widehat{x}_{\rm o}, r_{\rm c}) = \widehat{\alpha}_{\rm d}$.*

We can summarize our results as follows. Let $w_{\rm rsp}$ denote the minimum expenditure which is obtained as the solution of the RSP, eq.(11): $w_{\rm rsp}(p, r_{\rm c}, \widehat{\alpha}_{\rm d}) = p^T \widehat{x}_{\rm r}(p, r_{\rm c}, \widehat{\alpha}_{\rm d})$ where $\widehat{x}_{\rm r}(p, r_{\rm c}, \widehat{\alpha}_{\rm d})$ is the solution of the RSP. Proposition 2 relates $\widehat{x}_{\rm r}(p, r_{\rm c}, \widehat{\alpha}_{\rm d})$ to $\widehat{x}_{\rm o}(p, r_{\rm w}, w)$ (the solution of the OWP) as follows:

$$\widehat{x}_{\rm r}(p, r_{\rm c}, \widehat{\alpha}_{\rm d}) = \widehat{x}_{\rm o}[p, \underbrace{\mathcal{R}^*[\widehat{x}_{\rm r}, \mathcal{F}(\widehat{\alpha}_{\rm d}, \widetilde{f})]}_{r_{\rm w}}, w_{\rm rsp}(p, r_{\rm c}, \widehat{\alpha}_{\rm d})] \tag{24}$$

This relation explains the sense in which $\widehat{x}_{\rm r}$ is a **wealth-compensated demand.** By definition, the demand $\widehat{x}_{\rm r}$(p, $r_{\rm c}$, $\widehat{\alpha}_{\rm d}$) is evaluated with fixed critical reward $r_{\rm c}$ and fixed demanded robustness $\widehat{\alpha}_{\rm d}$. Moreover, eq.(24) shows that, as prices change, $\widehat{x}_{\rm r}(p, r_{\rm c}, \widehat{\alpha}_{\rm d})$ gives the consumption when the wealth $w_{\rm rsp}$(p, $r_{\rm c}$, $\widehat{\alpha}_{\rm d}$), is adjusted while keeping both $r_{\rm c}$ and $\widehat{\alpha}_{\rm d}$ constant. This parallels the sense in which the Hicksian demand function is a compensated demand [16, p.62]. This further strengthens the parallel between $\widehat{x}_{\rm r}$ and the Hicksian demand, since we already know that they both obey price-demand trade-off (proposition 1).

Let $\alpha_{\rm owp}(r_{\rm w}, w)$ denote the minimum $\widehat{\beta}$ obtained as the solution of the OWP: the lowest info-gap which facilitates windfall reward $r_{\rm w}$ given wealth w. Thus $\alpha_{\rm owp}(r_{\rm w}, w) = \widehat{\beta}(\widehat{x}_{\rm o}, r_{\rm w})$. Proposition 3 relates the solutions of the OWP and the RSP as:

$$\widehat{x}_{\rm o}(r_{\rm w}, w) = \widehat{x}_{\rm r}[\underbrace{\mathcal{R}_*[\widehat{x}_{\rm o}, \mathcal{F}(\alpha_{\rm owp}, \widetilde{f})]}_{r_{\rm c}}, \alpha_{\rm owp}(r_{\rm w}, w)] \tag{25}$$

Propositions 2 and 3, together with lemmas 1 and 2, also assert that:

$$\widehat{\alpha}(\widehat{x}_{\rm r}, \underbrace{\mathcal{R}_*[\widehat{x}_{\rm r}, \mathcal{F}(\widehat{\alpha}_{\rm d}, \widetilde{f})]}_{r_{\rm c}}) = \widehat{\beta}(\widehat{x}_{\rm r}, \underbrace{\mathcal{R}^*[\widehat{x}_{\rm r}, \mathcal{F}(\widehat{\alpha}_{\rm d}, \widetilde{f})]}_{r_{\rm w}}) \tag{26}$$

$$\widehat{\alpha}(\widehat{x}_{\rm o}, \underbrace{\mathcal{R}_*[\widehat{x}_{\rm o}, \mathcal{F}(\alpha_{\rm owp}, \widetilde{f})]}_{r_{\rm c}}) = \widehat{\beta}(\widehat{x}_{\rm o}, \underbrace{\mathcal{R}^*[\widehat{x}_{\rm o}, \mathcal{F}(\alpha_{\rm owp}, \widetilde{f})]}_{r_{\rm w}}) \tag{27}$$

Each of these relations establishes r_c and r_w values at which the robustness and opportunity functions obtain the same values when evaluated at the same level of demand.

Consider for instance eq.(26). When the consumer chooses the RSP-optimal demand $\widehat{x}_r$, then critical reward no less than r_c is guaranteed if the info-gap is no greater than $\widehat{\alpha}(\widehat{x}_r, r_c)$. Eq.(26) asserts that this is the lowest level of uncertainty at which reward as large as r_w is possible. In other words, eq.(26) identifies a situation in which survival (r_c) is guaranteed while windfall (r_w) is possible. A similar situation is identified by eq.(27).

Furthermore, it can be shown that $\widehat{\alpha}(x, r_c)$ decreases as r_c increases, while $\widehat{\beta}(x, r_w)$ increases as r_w increases [13]. Thus eq.(26) specifies the greatest feasible windfall r_w at which critical reward r_c is robustly guaranteed, when demand $\widehat{x}_r$ is chosen. Any greater value of r_w cannot be attained without increasing the info-gap above the value of robustness $\widehat{\alpha}(\widehat{x}_r, r_c)$. Conversely, r_c is the lowest value of critical survival-reward which can be robustly guaranteed while also enabling windfall as large as r_w, when choosing $\widehat{x}_r$. Any larger value of r_c would entail lower robustness and hence a lower feasible value of windfall reward r_w. Similar conclusions apply to eq.(27) when demand function $\widehat{x}_o$ is chosen.

The lower and upper reward functions are very commonly 'naturally ordered' (definition 8). We will explain that this property implies:

$$r_c \le r_w \tag{28}$$

for each of the r_c-r_w pairs in eqs.(26) and (27).

Definition 8 *Lower and upper reward functions* $\mathcal{R}_*(x, A)$ *and* $\mathcal{R}^*(x, A)$ *are* **naturally ordered** *if, for all* $x \in X$ *and for all* $f \in \Omega$*:*

$$\mathcal{R}_*(x, \{f\}) \le \mathcal{R}^*(x, \{f\}) \tag{29}$$

We have not assumed natural ordering of our reward functions, but it does in fact hold for the reward functions of eqs.(5) and (6). Natural ordering together with the monotonicity properties, eqs.(3) and (4), imply relation (28) for the r_c and r_w values in eqs.(26) and (27).

7 Examples

We now consider several simple examples of robustness functions and robust-satisficing demand functions obtained as solutions of the RSP.

7.1 Elasticity: 1

Consider non-negative choices, x_1 and x_2, for two commodities for which the reward function is $r(x, f) = f x_1 x_2$ where the fractional variation of the reward coefficient f is unknown:

$$\mathcal{F}(\alpha, \widetilde{f}) = \left\{ f : \left| \frac{f - \widetilde{f}}{\widetilde{f}} \right| \le \alpha \right\}, \quad \alpha \ge 0 \tag{30}$$

The known nominal coefficient $\widetilde{f}$ is positive.

The robustness of choices x_1 and x_2 is the greatest info-gap at which the reward is no less than the aspiration, r_c:

$$\widehat{\alpha}(x, r_c) = \max \left\{ \alpha : \min_{f \in \mathcal{F}(\alpha, \widetilde{f})} f x_1 x_2 \ge r_c \right\} \tag{31}$$

The robustness function becomes:

$$\widehat{\alpha}(x, r_c) = 1 - \frac{r_c}{\widetilde{f} x_1 x_2} \tag{32}$$

or zero if this expression is negative. If $r_c \ge 0$ then the robustness is no greater than unity. Hence the demanded robustness, $\widehat{\alpha}_d$, can also be no greater than unity.

The RSP, defined in eq.(11), can be re-arranged to the following non-linear optimization problem:

$$\min_{x \in X} p^T x \quad \text{subject to} \quad x_1 x_2 \ge \mu \tag{33}$$

where we have defined $\mu = \frac{r_c}{(1-\widehat{\alpha}_d)\widetilde{f}}$. The solution yields the following robust-satisficing demand function:

$$\widehat{x}_1 = \sqrt{\frac{\mu p_2}{p_1}}, \quad \widehat{x}_2 = \sqrt{\frac{\mu p_1}{p_2}} \tag{34}$$

We note that these demand functions are homogeneous of degree zero in the price. That is, for any $\theta \neq 0$:

$$\widehat{x}_i(\theta p) = \widehat{x}_i(p) \tag{35}$$

as is true of ordinary hicksian demand functions [17, pp.43].

We find the substitution matrix to be:

$$\frac{\partial \widehat{x}}{\partial p} = \frac{1}{2}\sqrt{\frac{\mu}{p_1 p_2}} \begin{pmatrix} -\frac{p_2}{p_1} & 1 \\ 1 & -\frac{p_1}{p_2} \end{pmatrix} \tag{36}$$

which is symmetric, negative semi-definite and has the price vector p in its null space, as is true of ordinary hicksian demand functions [17, pp.43–44]. The own-price elasticities are negative and the cross-price elasticities are positive. The elasticities increase in magnitude with increasing aspiration r_c, and with increasing demanded robustness $\widehat{\alpha}_d$ (both of which appear in the coefficient μ). The elasticities decrease in magnitude as the nominal return coefficient $\widetilde{f}$ increases. The substitution curve, $\widehat{x}_2$ vs. $\widehat{x}_1$, is a hyperbole: $\widehat{x}_1 \widehat{x}_2 = \mu$ for all positive levels of consumption.

7.2 Elasticity: 2

7.2.1 Formulation

Consider the choice vector $x \in \Re^N_+$ of non-negative elements, for which the reward function is $r(x, f) = x^T F x$ where the matrix F is uncertain and belongs to an interval-bound info-gap model:

$$\mathcal{F}(\alpha, \widetilde{f}) = \left\{ f : \left| f_{ij} - \widetilde{f}_{ij} \right| \le w_{ij}\alpha, \ i, j = 1, \ldots, N \right\}, \quad \alpha \ge 0 \tag{37}$$

Let $\widetilde{F}$ and W denote the known symmetric matrices of nominal reward coefficients $\widetilde{f}_{ij}$ and uncertainty weights w_{ij}. The uncertainty weights w_{ij} are all non-negative.

The robustness of choice x is the greatest info-gap at which the reward is no less than the aspiration r_c:

$$\widehat{\alpha}(x, r_c) = \max \left\{ \alpha : \min_{F \in \mathcal{F}(\alpha, \widetilde{f})} x^T F x \ge r_c \right\} \tag{38}$$

The lowest reward, up to uncertainty α, is:

$$\min_{F \in \mathcal{F}(\alpha, \widetilde{F})} x^T F x = x^T [\widetilde{F} - \alpha W] x \tag{39}$$

The reward-aspiration r_c is nominally feasible if:

$$x^T \widetilde{F} x \ge r_c \tag{40}$$

When this holds, the robustness $\widehat{\alpha}$ is infinite if $x^T W x = 0$, while $\widehat{\alpha}$ is finite otherwise. On the other hand, if condition (40) is violated, meaning that the reward-aspiration is infeasible, then the robustness is zero. In summary:

$$\widehat{\alpha}(x, r_{\rm c}) = \left\{ \begin{array}{cl} \infty & \text{if } \ x^T W x = 0 \ \text{ and } \ x^T \widetilde{F} x \geq r_{\rm c} \\ \dfrac{x^T \widetilde{F} x - r_{\rm c}}{x^T W x} & \text{if } \ x^T W x > 0 \ \text{ and } \ x^T \widetilde{F} x \geq r_{\rm c} \\ 0 & \text{if } \ x^T \widetilde{F} x < r_{\rm c} \end{array} \right. \tag{41}$$

7.2.2 Diagonal Case

Having derived the robustness function we now proceed to consider the solution of the RSP. The commodity vector x which solves the RSP is the robust-satisficing demand function, $\widehat{x}$.

Consider the special case that $\widetilde{F}$ and W are diagonal matrices of positive numbers:

$$\widetilde{F} = \text{diag}(\widetilde{f}_{11}, \ldots, \widetilde{f}_{NN}), \quad W = \text{diag}(w_{11}, \ldots, w_{NN}) \tag{42}$$

For simplicity consider only x-values for which $x^T W x > 0$ and $x^T \widetilde{F} x \geq r_{\rm c}$. The RSP can be formulated as:

$$\min_{x \geq 0} p^T x \quad \text{subject to} \quad x^T \underbrace{[\widetilde{F} - \widehat{\alpha}_{\rm d} W]}_{A} x = r_{\rm c} \tag{43}$$

which defines the matrix A, which we assume to be non-singular. The demand function becomes:

$$\widehat{x} = \sqrt{\frac{r_{\rm c}}{p^T A^{-1} p}} A^{-1} p \tag{44}$$

The matrix A is diagonal with elements $a_{ii} = \widetilde{f}_{ii} - \widehat{\alpha}_{\rm d} w_{ii}$. It is evident from the RSP that if $a_{ii} \leq 0$ for some i, then the corresponding robust-satisficing demand must be zero, $\widehat{x}_i = 0$, in order to minimize the expenditure $p^T x$. This means that if the uncertainty (w_{ii}) in commodity i, weighted by the aspiration for global robustness ($\widehat{\alpha}_{\rm d}$), exceeds the nominal reward ($\widetilde{f}_{ii}$) from commodity i, then the demand for that commodity vanishes. More generally we see how the RSP of eq.(43) balances reward-anticipation, uncertainty and robustness in formulating the robust-satisficing demand.

Let $\mathcal{I}$ be the set of indices of those commodities for which $a_{ii} > 0$. Let $\mathcal{I}_{-i}$ denote the set $\mathcal{I}$ after index i has been removed. The robust-satisficing demands are:

$$\widehat{x}_i = \sqrt{\frac{r_{\rm c}}{\displaystyle\sum_{j \in \mathcal{I}} p_j^2 / a_{jj}}} \frac{p_i}{a_{ii}}, \quad i \in \mathcal{I} \tag{45}$$

The own- and cross-price substitutions, for i, $j \in \mathcal{I}$, are:

$$\frac{\partial \widehat{x}_i}{\partial p_i} = \frac{\sqrt{r_{\rm c}}}{a_{ii}} \sum_{j \in \mathcal{I}_{-i}} p_j^2 / a_{jj} \left(\sum_{j \in \mathcal{I}} p_j^2 / a_{jj} \right)^{-3/2} \tag{46}$$

$$\frac{\partial \widehat{x}_i}{\partial p_j} = -\sqrt{r_{\rm c}} \frac{p_i p_j}{a_{ii} a_{jj}} \left(\sum_{j \in \mathcal{I}} p_j^2 / a_{jj} \right)^{-3/2}, \quad i \neq j \tag{47}$$

The own-price substitutions are all positive, while the cross-price substitutions are all negative.

The explanation of the signs of these substitutions lies in the linear-quadratic optimization of eq.(43), illustrated schematically in fig. 1. The curve is $x^T A x = r_{\rm c}$ drawn for arbitrary (not

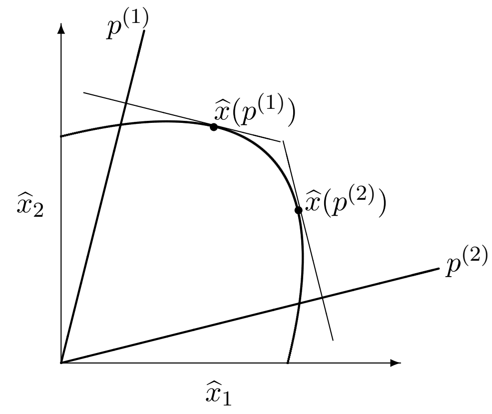

Figure 1: Illustration of price elasticities, eqs.(46) and (47).

necessarily diagonal) A. Consider first the price vector labelled $p^{(1)}$. The solution of the RSP is the smallest non-negative vector $\widehat{x}$ on a line perpendicular to $p^{(1)}$ and contained on the quadratic surface $x^T Ax = r_{\rm c}$. The solution is denoted $\widehat{x}(p^{(1)})$. Now consider the price vector $p^{(2)}$ for which the price of the first and second commodities have increased and decreased, respectively. The robust-satisficing demand function for $p^{(2)}$ is the point marked $\widehat{x}(p^{(2)})$: demand has increased for the first commodity and decreased for the second, as anticipated by eqs.(46) and (47).

It is evident that if we consider a third price vector, obtained by rotating $p^{(2)}$ further towards the $\widehat{x}_1$ axis, the "return" curvature of the quadratic surface, if it is strong enough, would eventually cause the demand function to display negative own-price substitutions and positive cross-price substitutions. However, this will not happen when A is diagonal, as in eqs.(46) and (47), since the quadratic surface is an ellipsoid with axes parallel to the coordinate axes and thus has no "return" curvature at all within the positive quadrant. However, if A is not diagonal it may define an ellipsoid tilted with respect to the axes as illustrated in fig. 1.

In short, the matrix $A = \widetilde{F} - \widehat{\alpha}_{\rm d} W$ controls the signs and magnitudes of the elements of the substitution matrix. $\widetilde{F}$ is the matrix of anticipated reward coefficients, $\widehat{\alpha}_{\rm d}$ is the demanded robustness and W is the matrix of uncertainty-weights in the info-gap model.

8 Summary and Conclusion

We have developed a theory of consumer demand which preserves the phenomenological features of classical demand, without the assumptions of rational-choice theory: complete, transitive preferences. We have explained that the info-gap optimization problems of robust satisficing (RSP) and opportune windfalling (OWP) correspond to the neoclassical optimization problems of expenditure minimization (EMP) and utility maximization (UMP), respectively. We have demonstrated that the consumer choice resulting from the RSP, $\widehat{x}_{\rm r}$, is the analog of the Hicksian demand function, while the solution of the OWP, $\widehat{x}_{\rm o}$, is analogous to the Walrasian demand. Specifically, $\widehat{x}_{\rm r}$ is a wealth-compensated demand function which obeys Walras' law and the law of price-demand trade-off.

The workhorses of info-gap analysis are the immunity functions: the robustness function $\widehat{\alpha}(x, r_{\rm c})$ and the opportunity function $\widehat{\beta}(x, r_{\rm w})$. These functions are derived from an info-gap model, $\mathcal{F}(\alpha, \widetilde{f})$, $\alpha \geq 0$, which is an unbounded family of nested sets of events, and which expresses the unbounded domain of pernicious as well as propitious possibilities entailed by the decision maker's incomplete knowledge. The immunities also depend upon the lower and upper reward functions, $\mathcal{R}_*(x, A)$ and $\mathcal{R}^*(x, A)$, which describe anticipated reward accruing from choice

x accompanied by uncertain environment A.

The robustness function $\widehat{\alpha}(x, r_{\rm c})$ assesses the degree of robustness of choice x to pernicious ambient uncertainty, when the decision maker aspires to satisfice at the reward level $r_{\rm c}$. The opportunity function $\widehat{\beta}(x, r_{\rm w})$ expresses the immunity of option x to the possible attainment of a great windfall reward $r_{\rm w}$. Both immunity functions depend upon aspirations, $r_{\rm c}$ and $r_{\rm w}$, which are not specified by the theory nor chosen *a priori* by the consumer. Once the aspiration levels $r_{\rm c}$ and $r_{\rm w}$ are chosen, then each immunity function generates a complete transitive preference ranking of the options. However, these preference rankings — based on robustness or opportunity — need not be consistent with one another, nor need they remain constant as aspirations change. Since wealth, price and other contextual factors will influence consumer aspiration, info-gap theory does not presume prior knowledge of preference, nor can it predict consumer choice. Even in this context, however, we have seen that the basic results of demand theory remain intact: complementarity of the robust-satisficing and the opportune-windfalling choice problems; Hicksian- and Walrasian-like demand functions derived from these consumer-choice optimization problems; wealth-compensated price-demand trade-off.

9 Appendix: Proofs

Proof of proposition 1. $\widehat{x}_{\rm r}(p, r_{\rm c}, \widehat{\alpha}_{\rm d})$ minimizes the expenditure with price vector p, so for any p':

$$p^T \widehat{x}_{\rm r}(p, r_{\rm c}, \widehat{\alpha}_{\rm d}) \le p^T \widehat{x}_{\rm r}(p', r_{\rm c}, \widehat{\alpha}_{\rm d}) \qquad (48)$$

Likewise, $\widehat{x}_{\rm r}(p', r_{\rm c}, \widehat{\alpha}_{\rm d})$ minimizes the expenditure with price vector p', so for any p:

$$p'^T \widehat{x}_{\rm r}(p, r_{\rm c}, \widehat{\alpha}_{\rm d}) \ge p'^T \widehat{x}_{\rm r}(p', r_{\rm c}, \widehat{\alpha}_{\rm d}) \qquad (49)$$

Subtracting (49) from (48) and re-arranging leads to the desired result, eq.(14). ■

Proof of lemma 1. (1) From the definition of the robustness function in eq.(7) and the continuity of $\mathcal{R}_*$ in α, and since x solves the RSP, we require that:

$$\mathcal{R}_*[x, \mathcal{F}(\widehat{\alpha}(x, r_{\rm c}), \widetilde{f})] \ge r_{\rm c} \qquad (50)$$

Suppose that:

$$\mathcal{R}_*[x, \mathcal{F}(\widehat{\alpha}(x, r_{\rm c}), \widetilde{f})] > r_{\rm c} \qquad (51)$$

By the info-gap axiom of nesting we have: $\alpha < \alpha'$ implies that $\mathcal{F}(\alpha, \widetilde{f}) \subset \mathcal{F}(\alpha', \widetilde{f})$. Since $\mathcal{R}_*(x, A)$ is monotonically decreasing in A (eq.(3)) and continuous in α, (51) implies that there is an $\alpha > \widehat{\alpha}(x, r_{\rm c})$ such that:

$$\mathcal{R}_*[x, \mathcal{F}(\alpha, \widetilde{f})] > r_{\rm c} \qquad (52)$$

which contradicts the definition of $\widehat{\alpha}(x, r_{\rm c})$. Hence

$$\mathcal{R}_*[x, \mathcal{F}(\widehat{\alpha}(x, r_{\rm c}), \widetilde{f})] = r_{\rm c} \qquad (53)$$

(2) Suppose that $\widehat{\alpha}(x, r_{\rm c}) > \widehat{\alpha}_{\rm d}$. Based on this supposition, and since $\widehat{\alpha}(x, r_{\rm c})$ is continuous in x, there is an $x' \in X$ such that $p^T x' < p^T x$ and $\widehat{\alpha}(x', r_{\rm c}) > \widehat{\alpha}_{\rm d}$. This contradicts the supposition of the lemma that x solves the RSP. Hence there is no such x'. Consequently $\widehat{\alpha}(x, r_{\rm c}) = \widehat{\alpha}_{\rm d}$. ■

Proof of lemma 2. (1) From the definition of the opportunity function in eq.(8) and the continuity of $\mathcal{R}^*$ in α, and since x solves the OWP, we require that:

$$\mathcal{R}^*[x, \mathcal{F}(\widehat{\beta}(x, r_{\rm w}), \widetilde{f})] \ge r_{\rm w} \qquad (54)$$

Suppose that:

$$\mathcal{R}^*[x, \mathcal{F}(\widehat{\beta}(x, r_{\mathrm{w}}), \widetilde{f})] > r_{\mathrm{w}} \tag{55}$$

By the info-gap axiom of nesting we have: $\alpha < \alpha'$ implies that $\mathcal{F}(\alpha, \widetilde{f}) \subset \mathcal{F}(\alpha', \widetilde{f})$. Since $\mathcal{R}^*(x, A)$ is monotonically increasing in A (eq.(4)) and continuous in α, (55) implies that there is an $\alpha < \widehat{\beta}(x, r_{\mathrm{w}})$ such that:

$$\mathcal{R}^*[x, \mathcal{F}(\alpha, \widetilde{f})] > r_{\mathrm{w}} \tag{56}$$

which contradicts the definition of $\widehat{\beta}(x, r_{\mathrm{w}})$. Hence

$$\mathcal{R}^*[x, \mathcal{F}(\widehat{\beta}(x, r_{\mathrm{w}}), \widetilde{f})] = r_{\mathrm{w}} \tag{57}$$

(2) Suppose that $p^T x < w$. Based on this supposition, and since $\widehat{\beta}(x, r_{\mathrm{w}})$ is non-satiated at x, there is an $x' \in X$ such that $\widehat{\beta}(x', r_{\mathrm{w}}) < \widehat{\beta}(x, r_{\mathrm{w}})$ and $p^T x' < w$. This contradicts the supposition of the lemma that x solves the OWP. Hence there is no such x'. Consequently $p^T x = w$. ∎

Proof of Proposition 2. $\widehat{\alpha}_{\mathrm{d}} = \widehat{\alpha}(\widehat{x}_{\mathrm{r}}, r_{\mathrm{c}})$ by lemma 1.

Suppose, contrary to the assertion of the proposition, that there is an $x'' \in X$ which solves the OWP with the specified w and r_{w}, so that $p^T x'' \leq w$, and for which:

$$\widehat{\beta}(x'', r_{\mathrm{w}}) < \widehat{\beta}(\widehat{x}_{\mathrm{r}}, r_{\mathrm{w}}) \tag{58}$$

which means that $\widehat{x}_{\mathrm{r}}$ does not solve the OWP. We will refer to this contradictory supposition as CS.

Given CS and the continuity of the opportunity function $\widehat{\beta}(x, r_{\mathrm{w}})$, we see that there is an x' such that:

$$\widehat{\beta}(x', r_{\mathrm{w}}) < \widehat{\beta}(\widehat{x}_{\mathrm{r}}, r_{\mathrm{w}}) \quad \text{and} \quad p^T x' < w \tag{59}$$

That is, the opportunity is better at x' than $\widehat{x}_{\mathrm{r}}$, and the expenditure with x' is strictly less than w.

By definition of the opportunity function:

$$\widehat{\beta}(x', r_{\mathrm{w}}) = \min \left\{ \alpha : \ \mathcal{R}^*[x', \mathcal{F}(\alpha, \widetilde{f})] \geq \underbrace{\mathcal{R}^*[\widehat{x}_{\mathrm{r}}, \mathcal{F}(\widehat{\alpha}_{\mathrm{d}}, \widetilde{f})]}_{r_{\mathrm{w}}} \right\} \tag{60}$$

and:

$$\widehat{\beta}(\widehat{x}_{\mathrm{r}}, r_{\mathrm{w}}) = \min \left\{ \alpha : \ \mathcal{R}^*[\widehat{x}_{\mathrm{r}}, \mathcal{F}(\alpha, \widetilde{f})] \geq \underbrace{\mathcal{R}^*[\widehat{x}_{\mathrm{r}}, \mathcal{F}(\widehat{\alpha}_{\mathrm{d}}, \widetilde{f})]}_{r_{\mathrm{w}}} \right\} = \widehat{\alpha}_{\mathrm{d}} \tag{61}$$

Equality to $\widehat{\alpha}_{\mathrm{d}}$ in eq.(61) arises as follows. Because $\widehat{\alpha}_{\mathrm{d}}$ appears in the expression for r_{w} in (61), it is evident from (61) that $\widehat{\beta}(\widehat{x}_{\mathrm{r}}, r_{\mathrm{w}}) \leq \widehat{\alpha}_{\mathrm{d}}$. From the nesting axiom of info-gap models we see that $\alpha < \widehat{\alpha}_{\mathrm{d}}$ implies that $\mathcal{F}(\alpha, \widetilde{f}) \subset \mathcal{F}(\widehat{\alpha}_{\mathrm{d}}, \widetilde{f})$. Strict monotonicity of $\mathcal{R}^*(x, A)$ in α implies $\mathcal{R}^*[\widehat{x}_{\mathrm{r}}, \mathcal{F}(\alpha, \widetilde{f})] < \mathcal{R}^*[\widehat{x}_{\mathrm{r}}, \mathcal{F}(\widehat{\alpha}_{\mathrm{d}}, \widetilde{f})]$ for $\alpha < \widehat{\alpha}_{\mathrm{d}}$. Hence $\widehat{\beta}(\widehat{x}_{\mathrm{r}}, r_{\mathrm{w}})$ cannot be less than $\widehat{\alpha}_{\mathrm{d}}$. Therefore $\widehat{\beta}(\widehat{x}_{\mathrm{r}}, r_{\mathrm{w}}) = \widehat{\alpha}_{\mathrm{d}}$.

Relations (59)–(61), together with monotonicity of $\mathcal{R}^*(x, A)$ in α, imply:

$$\mathcal{R}^*[x', \mathcal{F}(\widehat{\alpha}_{\mathrm{d}}, \widetilde{f})] > \mathcal{R}^*[\widehat{x}_{\mathrm{r}}, \mathcal{F}(\widehat{\alpha}_{\mathrm{d}}, \widetilde{f})] \tag{62}$$

By similar ordering of $\mathcal{R}_*$ and $\mathcal{R}^*$ this implies:

$$\mathcal{R}_*[x', \mathcal{F}(\widehat{\alpha}_{\mathrm{d}}, \widetilde{f})] > \mathcal{R}_*[\widehat{x}_{\mathrm{r}}, \mathcal{F}(\widehat{\alpha}_{\mathrm{d}}, \widetilde{f})] \tag{63}$$

By definition of the robustness function:

$$\widehat{\alpha}(x', r_c) = \max\left\{\alpha:\ \mathcal{R}_*[x', \mathcal{F}(\alpha, \widetilde{f})] \geq \underbrace{\mathcal{R}_*[\widehat{x}_r, \mathcal{F}(\widehat{\alpha}_d, \widetilde{f})]}_{r_c}\right\} \tag{64}$$

The identity to r_c results from lemma 1.

Lemma 1 also implies that $\widehat{\alpha}(\widehat{x}_r, r_c) = \widehat{\alpha}_d$. Relations (63) and (64), together with monotonicity of $\mathcal{R}_*(x, A)$ in the uncertainty sets A, relation (3), and continuity in α, imply:

$$\widehat{\alpha}(x', r_c) > \widehat{\alpha}_d = \widehat{\alpha}(\widehat{x}_r, r_c) \tag{65}$$

Since $p^T x' < w$ (eq.(59)), while (by definition of w) $p^T \widehat{x}_r = w$, we see from (65) that CS implies that $\widehat{x}_r$ is **not** a solution of the RSP. This contradicts the condition of the proposition, so CS is false. Hence there is no x'' satisfying relation (58). We conclude that $\widehat{x}_r$ solves the OWP: it minimizes the opportunity function and $p^T \widehat{x}_r = w$. ■

Proof of Proposition 3. Lemma 2 implies $p^T \widehat{x}_o = w$.

Suppose, contrary to the assertion of the proposition, that there is an $x'' \in X$ which solves the RSP with the specified r_c and $\widehat{\alpha}_d$, so that $\widehat{\alpha}(x'', r_c) \geq \widehat{\alpha}_d$ and for which:

$$p^T x'' < p^T \widehat{x}_o \tag{66}$$

which means that $\widehat{x}_o$ does not solve the RSP. We will refer to this contradictory supposition as CS.

Given CS and the non-satiation of the robustness function $\widehat{\alpha}(x, r_c)$, we see that there is an x' such that:

$$p^T x' < p^T \widehat{x}_o \quad \text{and} \quad \widehat{\alpha}(x', r_c) > \widehat{\alpha}_d \tag{67}$$

That is, the expenditure is lower at x' than $\widehat{x}_o$, and the robustness with x' is strictly greater than $\widehat{\alpha}_d$.

By definition of the robustness function:

$$\widehat{\alpha}(x', r_c) = \max\left\{\alpha:\ \mathcal{R}_*[x', \mathcal{F}(\alpha, \widetilde{f})] \geq \underbrace{\mathcal{R}_*[\widehat{x}_o, \mathcal{F}(\widehat{\alpha}_d, \widetilde{f})]}_{r_c}\right\} \tag{68}$$

and

$$\widehat{\alpha}(\widehat{x}_o, r_c) = \max\left\{\alpha:\ \mathcal{R}_*[\widehat{x}_o, \mathcal{F}(\alpha, \widetilde{f})] \geq \underbrace{\mathcal{R}_*[\widehat{x}_o, \mathcal{F}(\widehat{\alpha}_d, \widetilde{f})]}_{r_c}\right\} = \widehat{\alpha}_d \tag{69}$$

Equality to $\widehat{\alpha}_d$ in eq.(69) arises as follows. It is evident from (69) that $\widehat{\alpha}(\widehat{x}_o, r_c) \geq \widehat{\alpha}_d$. From the nesting axiom of info-gap models we see that $\widehat{\alpha}_d < \alpha$ implies that $\mathcal{F}(\widehat{\alpha}_d, \widetilde{f}) \subset \mathcal{F}(\alpha, \widetilde{f})$. Strict monotonicity of $\mathcal{R}_*(x, A)$ in α implies $\mathcal{R}_*[\widehat{x}_o, \mathcal{F}(\alpha, \widetilde{f})] < \mathcal{R}_*[\widehat{x}_o, \mathcal{F}(\widehat{\alpha}_d, \widetilde{f})]$ for $\widehat{\alpha}_d < \alpha$. Hence $\widehat{\alpha}(\widehat{x}_o, r_c)$ cannot be greater than $\widehat{\alpha}_d$. Therefore $\widehat{\alpha}(\widehat{x}_o, r_c) = \widehat{\alpha}_d$.

Relations (67)–(69), together with the monotonicity of $\mathcal{R}_*(x, A)$ in α, imply:

$$\mathcal{R}_*[x', \mathcal{F}(\widehat{\alpha}_d, \widetilde{f})] > \mathcal{R}_*[\widehat{x}_o, \mathcal{F}(\widehat{\alpha}_d, \widetilde{f})] \tag{70}$$

By similar ordering of $\mathcal{R}_*$ and $\mathcal{R}^*$:

$$\mathcal{R}^*[x', \mathcal{F}(\widehat{\alpha}_d, \widetilde{f})] > \mathcal{R}^*[\widehat{x}_o, \mathcal{F}(\widehat{\alpha}_d, \widetilde{f})] \tag{71}$$

By definition of the opportunity function:

$$\widehat{\beta}(x', r_{\rm w}) = \min \left\{ \alpha : \ \mathcal{R}^*[x', \mathcal{F}(\alpha, \widetilde{f})] \geq \underbrace{\mathcal{R}^*[\widehat{x}_{\rm o}, \mathcal{F}(\widehat{\alpha}_{\rm d}, \widetilde{f})]}_{r_{\rm w}} \right\} \tag{72}$$

The identity of $r_{\rm w}$ in (72) results from lemma 2.

By definition, $\widehat{\alpha}_{\rm d} = \widehat{\beta}(\widehat{x}_{\rm o}, r_{\rm w})$. Relations (71) and (72), together with the monotonity of $\mathcal{R}^*(x, A)$ in the sets A, relation (4), and continuity in α, imply:

$$\widehat{\beta}(x', r_{\rm w}) < \widehat{\alpha}_{\rm d} = \widehat{\beta}(\widehat{x}_{\rm o}, r_{\rm w}) \tag{73}$$

Since $p^T x' < p^T \widehat{x}_{\rm o}$ (from eq.(67)), and since $p^T \widehat{x}_{\rm o} = w$, (73) contradicts the supposition of the proposition that $\widehat{x}_{\rm o}$ solves the OWP. Hence there cannot be such an x' and the CS is false. Hence $\widehat{x}_{\rm o}$ solves the RSP. ■

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