Up-Dating a Linear-Stiffness Model with Uncertain Non-Linearity

Yakov Ben-Haim
Technion—Israel Institute of Technology
yakov@technion.ac.il, info-gap.com

The problem. We will use measurements to estimate the stiffness, k, of a 1-dimensional spring, where force f is assumed to be related to displacement x by a linear model: $f = kx$. However, we suspect that the spring may actually have a cubic non-linearity: $f = kx + k_3x^3$. The non-linear stiffness k_3 is highly uncertain, and for practical reasons we do not include this non-linearity in the up-dated model. We will describe an info-gap robust-satisficing approaching to up-dating the linear model given measurements, and subject to the uncertain non-linearity. More advanced analysis is available in the references cited below.

Mean squared estimates. Our force-deflection measurements are (x_i, f_i), $i = 1, \ldots, n$. For any choice of the linear stiffness, k, the mean squared error (MSE) of the linear model is:

$$S = \frac{1}{n}\sum_{i=1}^{n}(f_i - kx_i)^2 \tag{1}$$

The value of k which minimizes this MSE is denoted $\widehat{k}$.

The actual MSE, accounting for the uncertain non-linearity, is:

$$S(k, k_3) = \frac{1}{n}\sum_{i=1}^{n}(f_i - kx_i - k_3x_i^3)^2 \tag{2}$$

Robustness. We wish to choose the linear stiffness, k, so that the actual MSE is adequately small and so that we are robust to the uncertain non-linearity.

Consider a uniform-bound info-gap model for uncertainty in the cubic term:

$$\mathcal{U}(h) = \{k_3 : \ |k_3| \le h\}, \quad h \ge 0 \tag{3}$$

This is an unbounded family of nested sets of k_3 values. There is no probabilistic information and no known worst case.

The robustness of the linear model with coefficient k is the greatest horizon of uncertainty, h, up to which the actual mean squared error does not exceed S_c:

$$\widehat{h}(k, S_c) = \max\left\{h : \ \left(\max_{k_3\in\mathcal{U}(h)} S(k, k_3)\right) \le S_c\right\} \tag{4}$$

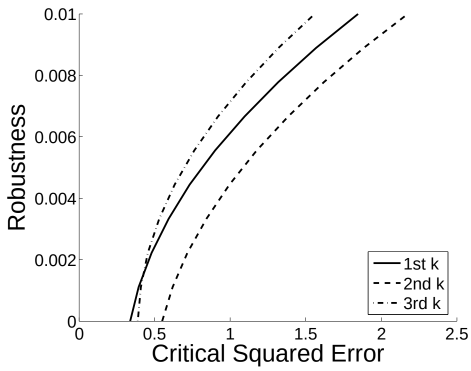

Figure 1: Robustness curves. $\widehat{k} = 1.8681$ (solid, least squares value), $k = 1.75$ (- -), $k = 1.81$ (-.). Data: $x^T = (1, 2, 3, 4, 5, 6)$, $f^T = (2.0, 3.6, 6.6, 7.8, 9.6, 10.3)$.

[0]\website\igt\simple-examples\model-update-uncer-nonlin01.tex 26.3.2009

Fig. 1 shows robustness curves for three different choices of the linear stiffness coefficient. The MSE optimal coefficient is $\widehat{k} = 1.8681$, whose robustness curve (solid line) sprouts from the S_c-axis to the left of all other curves. However, the robustness is zero for S_c values on the horizontal axis. Furthermore, the robustness curve for $k = 1.81$ crosses the MSE optimal curve at low robustness. This implies that $k = 1.81$ is a more reliable choice for the stiffness of the linear model in light of the uncertain non-linearity. The robustness curve for $k = 1.75$ lies below the other curves and thus $k = 1.75$ would never be chosen.

Sources.

◦ Yakov Ben-Haim, 2009, Up-Dating a Linear System with Model Uncertainty: An Info-Gap Approach, working paper.

◦ Yakov Ben-Haim and Scott Cogan, 2009, Linear Bounds on an Uncertain Non-Linear Oscillator: An Info-Gap Approach, working paper.