Structural Reliability with Uncertain Probability

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We consider the prototype of a wide range of structures: the linear spring, for which the force f is linearly related to the deflection x:

$$f = kx \tag{1}$$

It is necessary to choose (or to certify the choice of) the stiffness k such that the deflection x is acceptably small, subject to uncertain load f. The engineering requirement is that the magnitude of the deflection be no greater than a positive critical value $x_{\rm c}$:

$$|x| \le x_{\rm c} \tag{2}$$

Furthermore, we require that the probability of failure not exceed $P_{\rm c}$. Denoting the probability density function (pdf) of the load by $p(f)$, the probability of failure is:

$$P_{\rm f}(p) = \int_{-\infty}^{-kx_{\rm c}} p(f)\,{\rm d}f + \int_{kx_{\rm c}}^{\infty} p(f)\,{\rm d}f \tag{3}$$

Our probabilistic performance requirement is:

$$P_{\rm f}(p) \le P_{\rm c} \tag{4}$$

We suppose that we have an estimated pdf for the load, $\widetilde{p}(f)$. However, this density is uncertain, especially in its tails where failure occurs, so the evaluation of $P_{\rm f}(\widetilde{p})$ is also uncertain. Let $\mathcal{U}(h)$ denote an info-gap model for uncertainty in the estimated pdf of the load:

$$\mathcal{U}(h) = \left\{ p(f) :\ p(f) \ge 0,\ \int_{-\infty}^{\infty} p(f)\,{\rm d}f = 1,\ |p(f) - \widetilde{p}(f)| \le h\widetilde{p}(f) \right\}, \quad h \ge 0 \tag{5}$$

The robustness of design k is the greatest horizon of uncertainty h in the estimated pdf, $\widetilde{p}(f)$, up to which all pdfs indicate acceptably low probability of failure:

$$\widehat{h}(k, P_{\rm c}) = \max \left\{ h :\ \left(\max_{p \in \mathcal{U}(h)} P_{\rm f}(p) \right) \le P_{\rm c} \right\} \tag{6}$$

A large value of $\widehat{h}(k, P_{\rm c})$ indicates that the estimated probability of failure of the system with stiffness k is immune to errors in the estimated pdf, while a low value of $\widehat{h}(k, P_{\rm c})$ indicates that the estimated reliability of the system is vulnerable to error in $\widetilde{p}(f)$. $\widehat{h}(k, P_{\rm c})$ is the robustness (to uncertainty in the pdf) of the probabilistic estimate of the system reliability.

Let us assume that the estimated probability of failure, $P_{\rm f}(\widetilde{p})$, is less than 1/2; certainly not an uncommon situation. In this case we find that the robustness is:

$$\widehat{h}(k, P_{\rm c}) = \left\{ \begin{array}{cl} 0 & \text{if } P_{\rm c} \le P_{\rm f}(\widetilde{p}) \\ \dfrac{P_{\rm c}}{P_{\rm f}(\widetilde{p})} - 1 & \text{else} \end{array} \right. \tag{7}$$

Eq.(7) shows the usual trade-off between robustness and performance: small failure probability, $P_{\rm c}$, entails low robustness, $\widehat{h}(k, P_{\rm c})$, as illustrated in fig. 1. Furthermore the robustness is zero when the required probability of failure, $P_{\rm c}$, is less than or equals the estimated probability of failure, $P_{\rm f}(\widetilde{p})$.

Consider two different stiffnesses, $k > k'$, for which the former has lower estimated probability of failure:

$$P_{\rm f}(\widetilde{p}, k) < P_{\rm f}(\widetilde{p}, k') \tag{8}$$

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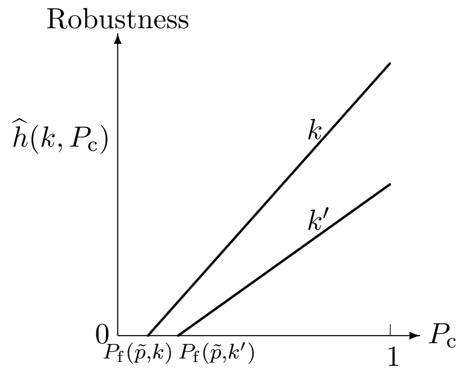

Figure 1: Robustness curves for structural reliability with uncertain pdf, eq.(7).

This, by itself, is not a particularly strong recommendation for choosing k over k', because the robustness is precisely zero for $P_c = P_f(\widetilde{p}, k)$. However, the robustness curve for k is steeper than for k', implying that the performance-cost of increasing the robustness is lower with k than with k'. This, combined with eq.(8), means that k is more reliable than k' in achieving P_c, for any $P_c > P_f(\widetilde{p}, k)$. If k is adequately reliable depends on the judgment that adequately large robustness, $\widehat{h}(k, P_c)$, is obtained at adequately small probability of failure, P_c.

Source.

◦ Yakov Ben-Haim, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press, London, section 3.2.3.