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# USABILITY OF MATHEMATICAL MODELS IN MECHANICAL DECISION PROCESSES

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Mathematical models of mechanical systems are developed to assist decision making for design, dynamic performance, structural interfacing, reliability, etc. Model updating is a costly and time-consuming task, so it is important to evaluate rigorously the quality of a model with respect to the decisions which rely upon it. The measure of model usability is that a model is usable to the extent that decisions based on the model are robust to the associated uncertainties. The analysis of robustness employs convex models of uncertainty, which are particularly suited to situations in which prior information about the uncertainties is severely limited. Several examples of the analysis of model usability for mechanical decisions are considered, including a simple harmonic oscillator, a simple non-linear system, and a multidimensional linear frequency response model for predicting dynamic stability.



## 1. INTRODUCTION

The widespread use of mathematical models to simulate the behaviour of complex mechanical structures naturally implies the need for some kind of quality control. All mathematical models are necessarily simplifications of a messier reality and it is important to study the implications of this approximation if the adequacy of the model for a given end use is to be judged. Not only must a model be verified and validated against empirical data, but it must also be evaluated with respect to its end use [1].

The mechanical design process implies the application of a design methodology to a set of objectives: reliability, ergonomy, minimal manufacturing and maintenance costs, etc. The designer proceeds through a hierarchy of decisions (dimensioning and materials selection, tolerance specifications, choosing preventive maintenance schedules, and so on) whose respective outcomes are intended to insure that these objectives are met. In the end, one is solely interested in making reliable decisions and the use of mathematical models is simply a means towards this end. In other words, if the designer decides to modify property $P$ in order to satisfy conditions $C$ on the basis of a model $M$, then it is desirable that, even if the predictions of model $M$ were incorrect in the worst possible way, this would not affect the outcome of the decision. This paper develops a method for evaluating the usability of a model which does essentially that, exploiting the concept of robust

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reliability [2, 3]. The degree of usability of a model will be measured by the robustness-to-uncertainties of decisions based on the model.

Section 2 presents a rigorous generic definition of robust model usability. Illustrative one-dimensional linear as well as non-linear examples are discussed in Section 3. Section 4 evaluates the degree of usability of a multidimensional frequency response model for predicting the dynamic stability of a structure.

## 2. ROBUST MODEL USABILITY

The measure of model usability developed here is: a model is usable to the extent that decisions based on the model are robust to the associated uncertainties. Usability is a matter of degree, and usability depends on the intended application of the model. One model is more usable than another when decisions based on the first model are more robust than decisions based on the second, and the degree of greater usability is measured by the degree of greater robustness. Also, as Natke [1] has stressed, the usability of a model depends on the purpose to which the model is applied, namely, the decisions which are based on the model. A model may be very useful for one decision but not at all for another.

Three components underlie the robust measure of model usability: (1) a mathematical model of the system: (2) a decision algorithm; and (3) a model of associated uncertainties.

The model of the system is denoted $S(v)$ and depends on various unknown parameters and functions, $v$, which may be stiffnesses, geometrical dimensions, input functions, and so on. There may also be known parameters in the model as well as design variables.

The decision algorithm 'operates' on the model and $D[S(v)]$ takes a value which may be a number or a linguistic variable. While decision algorithms come in a myriad of forms, this paper refers to 'bottom line' decisions which come, possibly, at the end of a sequence of prior decisions. These final decisions are often in a linguistic form and establish a go/no-go status of the system. The decision takes different forms in different applications, but typically the algorithm establishes whether or not the system, as represented by the model, satisfies some array of requirements for a given choice of $v$. For instance, the algorithm may determine if the maximum dynamic response is less than a critical value, in which case $D[S(v)]$ takes the value 'yes', and otherwise 'no'. Or, another typical decision algorithm selects one of a collection of alternative values for particular design variables in order to satisfy some performance requirements.

The uncertainty of the variables $v$ is represented by a convex model of uncertainty, which is denoted at present by $\mathscr{V}(\alpha)$. A convex model is a family of nested sets, $\mathscr{V}(\alpha)$ for $\alpha \geqslant 0$, which expands as the uncertainty parameter $\alpha$ grows, where $\alpha$ is a non-negative real number. Convex models of uncertainty have been described extensively elsewhere [3–5], and are discussed later on.

The robustness of the decision, which we propose as the measure of usability of the model, is the greatest value of the uncertainty parameter $\alpha$ for which the decision is the same for all events, in the convex model $\mathscr{V}(\alpha)$. Formally, the robustness is often expressed as follows:

$$\hat{\alpha} = \sup \{\alpha\colon D[S(v)] = \text{constant, for all } v \in \mathscr{V}(\alpha)\} \tag{1}$$

In other words, the robustness, $\hat{\alpha}$, is the supremum of the set of $\alpha$-values for which the decision is constant, and independent of $\alpha$, for all values of $v$ in $\mathscr{V}(\alpha)$. (In the examples, it is seen that modifications of this form are sometimes useful.) When $\hat{\alpha}$ is large, then the decision is stable with respect to the uncertainties. This means that the decision is the same over a wide range of values for the uncertain variables $v$. On the other hand, if $\hat{\alpha}$ is small then the decision is fragile with respect to the uncertainty: small variation of $v$ can lead to a change in the decision. If $\hat{\alpha}$ is large the decision is robust, insensitive to uncertainties, and the model can be reliably used for making the decision. If $\hat{\alpha}$ is small, then the model cannot be relied upon to yield consistent decisions.

Let us examine further why decision-robustness to uncertainty is a relevant measure of model usability. Uncertainties may occur in the operating environment of the system, for example load uncertainties, as well as in the properties of the system itself, such as material or geometrical uncertainties. In both cases, it is assumed that the uncertainty models are sufficiently 'rich' to encompass a realistic representation of the actual eventualities. Regarding environmental uncertainty this means that the convex model represents a realistic range of operational conditions. With regard to system uncertainties, it is assumed that the uncertainty model includes a realistic model within its domain, even though the nominal model upon which the decision is based may differ from more realistic models. Robustness to environmental uncertainty means that the decision is relevant to the full range of environmental contingencies, which is realistically represented by the uncertainty model. Robustness to system uncertainty means that the decision which is based on the nominal model is the same as the decision which would be obtained from a more realistic model, whose identity is not known but which is contained in the model of system uncertainty. This is discussed further in example 5.

Sometimes the convex model is centred at a 'nominal', 'typical' or 'design-base' value $\bar{v}$ of the uncertain quantity. The convex model is then denoted $\mathscr{V}(\alpha, \bar{v})$, for $\alpha \geqslant 0$, indicating a family of nested sets which are centred around $\bar{v}$. The robustness, equation (1), can then be written as:

$$\hat{\alpha} = \sup \{\alpha \colon D[S(v)] = D[S(\bar{v})] \quad \text{for all} \;\; v \in \mathscr{V}(\alpha, \bar{v})\} \tag{2}$$

Written in this form $\hat{\alpha}$ can be interpreted as the greatest value of the uncertainty parameter for which a decision obtained from the design-base value $\bar{v}$ is the same as the decision obtained from any other realisation of $v$ in the convex model. If $\hat{\alpha}$ is large then the design-base decision is 'correct' for any realization over a wide range of deviation from the nominal system, while if $\hat{\alpha}$ is small then even minor deviations of the system from its typical structure can cause an error in the nominal decision.

Concerning desirable magnitudes of $\hat{\alpha}$ it is clear that 'big is better'. But what are 'small' and 'large' values of $\hat{\alpha}$? How large is large enough? What increments in $\hat{\alpha}$ are significant? This is discussed in some of the examples that follow.

## 3. ONE-DIMENSIONAL EXAMPLES

### 3.1. EXAMPLE 1: LINEAR VIBRATION

Consider an undamped one-dimensional linear oscillator subject to uncertain input: $m\ddot{x}(t) + kx(t) = u(t)$ starting from zero initial conditions $x(0) = \dot{x}(0) = 0$. The system is 'acceptable' if the magnitude of the displacement $x$ remains less than the critical value $x_{cr}$ throughout a duration $T$. The uncertain input belongs to an 'energy-bound' convex model.

$$\mathscr{U}(\alpha) = \left\{ u(t) \colon \frac{1}{T} \int_0^T u^2(t)\, \mathrm{d}t \leqslant \alpha^2 \right\} \tag{3}$$

The model of the system is the differential oscillator equation, the uncertainty model is

$\mathscr{U}(\alpha)$, and the decision algorithm returns a value of 'yes' or 'no' depending on whether or not $|x(t)| \leqslant x_{cr}$ for all $0 \leqslant t \leqslant T$. The decision algorithm is formally expressed:

$$D(\alpha) = \text{yes} \quad \text{if and only if} \quad \max_{u \in \mathscr{U}(\alpha)} |x(t)| \leqslant x_{\max} \text{ for all } 0 \leqslant t \leqslant T \tag{4}$$

As described elsewhere [3, 5], Schwarz's inequality can be employed to establish the least upper bound of $x$ as $u$ varies on $\mathscr{U}(\alpha)$:

$$\hat{x}(t) = \frac{\alpha\sqrt{t}}{m\omega} \underbrace{\sqrt{\int_0^t \sin^2 \omega\tau \, \mathrm{d}\tau}}_{\sigma(t)} \tag{5}$$

which defines the quantity $\sigma(t)$. Note that $\sigma(t)$ increases monotonically with $t$. The natural frequency of the oscillator is $\omega = \sqrt{k/m}$.

The greatest value of uncertainty parameter for which the acceptability requirement is satisfied for all input functions in $\mathscr{U}(\alpha)$, is found by equating $\hat{x}(T)$ to the critical value of displacement and solving for $\alpha$:

$$\frac{\alpha\sigma(T)\sqrt{T}}{m\omega} = x_{cr} \Rightarrow \hat{\alpha} = \frac{x_{cr}\, m\omega}{\sigma(T)\sqrt{T}} \tag{6}$$

This is the robustness of the decision. If $\hat{\alpha}$ is large then the model is reliably used to decide whether the system, subject to uncertain input, is acceptable. If $\hat{\alpha}$ is small then the decision oscillates as the uncertain input varies over a small range of values. The formal expression of the robustness in this example is:

$$\hat{\alpha} = \sup\{\alpha : D(\alpha) = \text{yes}\} \tag{7}$$

The calibration of the robustness, $\hat{\alpha}$, to establish a scale from 'small' to 'large', can be approached in various ways. Two methods are discussed in [3] and an additional procedure is presented here. It is important to recognise that different calibrations may lead to different interpretations of the robustness. Consequently, calibration must be done with care, and with a clear view of the application at hand.

The uncertainty parameter has units of force, so $\hat{\alpha}x_{cr}$ is an energy. One way to calibrate the robustness is to compare $\hat{\alpha}x_{cr}$, thought of as a maximum input energy, against a typical dynamical energy of the oscillator. For example, the maximum strain energy of displacement of the oscillator is $\frac{1}{2}kx_{cr}^2$. 'Large' values of $\hat{\alpha}x_{cr}$ are much greater than this strain energy, while 'small' values of $\hat{\alpha}x_{cr}$ are much less. The dimensionless index of the robustness is:

$$\frac{\hat{\alpha}x_{cr}}{\frac{1}{2}kx_{cr}^2} = \frac{2m\omega}{k\sigma(T)\sqrt{T}} \tag{8}$$

The robustness if 'small', 'moderate' or 'large' if this dimensionless quantity is $\ll 1$, $\approx 1$ or $\gg 1$, respectively.

### 3.2. EXAMPLE 2: EXAMPLE 1 CONTINUED, CALIBRATION BY CONSEQUENCE SEVERITY

The robust usability $\hat{\alpha}$ is a function of properties of the system, as expressed by the model. The question often arises: what changes in the system, and consequently in the model, would make the decision substantially more reliable? For instance, equation (6) shows that any increase in $k$ (and hence in $\omega$) results in an increase in $\hat{\alpha}$. The question is, how large a stiffness change results in a significant change in robustness?

This requires subjective calibration of the robustness based on engineering judgement. Changes in $k$ and $\hat{\alpha}$ must be linked with some scale which can be interpreted in subjective or intuitive terms of 'small' or 'large'. The method of 'consequence severity' [3] is used here to calibrate increments of $\hat{\alpha}$ resulting from increments in $k$.

Our calibration of $\hat{\alpha}$ is based on a prior subjective interpretation of the failure threshold, $x_{cr}$. The system 'fails' if the absolute displacement exceeds the threshold, but it is a matter of engineering judgement to choose a value for $x_{cr}$. A large value of $x_{cr}$ defines failure to occur only with the onset of severe consequences, while a small value of $x_{cr}$ identifies failure even when only small impact on the system results. Engineering experience with the system may enable the analyst to assess what magnitudes of deflection correspond to minor or major consequences for the system. For an antenna deployed in space, for instance, vibrations of 1.0 mm may be 'large' while 0.1 mm may be 'small'. In contrast, for a heavy-duty crane, 'large' and 'small' vibrations may be 1.0 m and 0.1 m respectively. When this sort of judgement is available, it forms the basis of a subjective calibration of the robustness, as is now demonstrated.

First of all note that $\sigma(T) \approx \sqrt{T/2}$ if $T \gg 1/4\omega$. Then, recalling $\omega = \sqrt{k/m}$, the robustness can be written $\hat{\alpha} \approx \sqrt{2mk}x_{cr}/T$. Plotting $\hat{\alpha}$ versus $x_{cr}$ obtains the straight line in Fig. 1. $x_{cr,2}$ is the current threshold value upon which the decision is based and it represents moderate severity. Two other threshold values are chosen: $x_{cr,1}$ corresponds to low severity at the onset of failure, while $x_{cr,3}$ corresponds to high severity. Plotting $\hat{\alpha}$ versus the stiffness leads to the parabolic curve in Fig. 1. The straight curve is plotted for $k = k_2$, while the parabola is based on $x_{cr} = x_{cr,2}$.

In comparing these two curves, how large a change in stiffness induces a significant change in robustness? $\hat{\alpha}_2$ is the robustness of the decision based on the current model values, and one wishes to know how large a change in $k$ is needed to change $\hat{\alpha}$ meaningfully. At high failure severity ($x_{cr} = x_{cr,3}$), the robustness is $\hat{\alpha}_3$. It is reasonable to view $\hat{\alpha}_2$ and $\hat{\alpha}_3$ as 'substantially' different since they are the acceptable levels of input uncertainty which correspond to substantially different levels of failure severity. Following the thin line to the right and down in Fig. 1 leads to the value of stiffness which, with moderate failure severity ($x_{cr} = x_{cr,2}$), causes the substantially greater robustness value $\hat{\alpha}_3$. It is concluded that

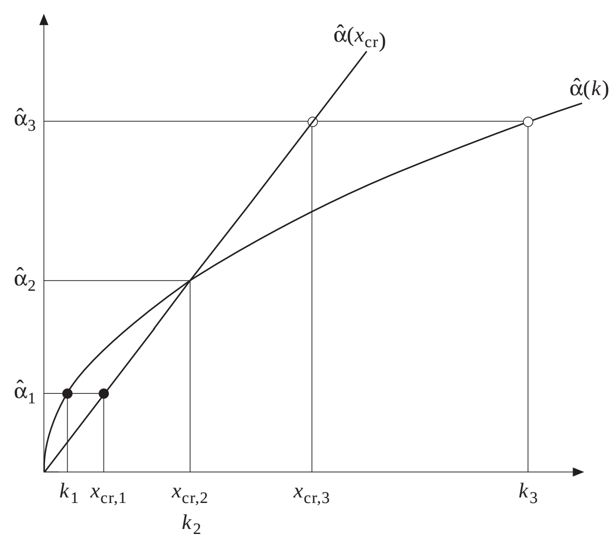


Figure 1. Calibrating stiffness and robustness changes in terms of failure-consequence severity.

the change in stiffness from $k_2$ to $k_3$ results in a significant increase in robustness of the decision. Elementary manipulations show that $k_2$ and $k_3$ are related as:

$$k_3 = k_2\left(1 + \frac{x_{\mathrm{cr},3} - x_{\mathrm{cr},2}}{x_{\mathrm{cr},2}}\right)^2 \tag{9}$$

Similarly, $\hat{\alpha}_1$ is a substantially lower robustness than $\hat{\alpha}_2$ since $x_{\mathrm{cr},1}$ and $x_{\mathrm{cr},2}$ represent low and moderate severities, respectively. Following the thin line in Fig. 1 leads to the stiffness value $k_1$ which, at moderate failure severity, generates low robustness $\hat{\alpha}_1$.

In this manner the model and system changes needed to implement substantial changes in the usability of the model for robust decisions have been established.

### 3.3. EXAMPLE 3: NON-LINEAR ELASTIC DISPLACEMENT

Joints and other structural components often display complex non-linear elastic behaviour which is difficult to model accurately. An acceptance test of a joint element can be based on its mechanical model. An idealised non-linear joint element is considered for which a decision must be made whether to accept the joint for use, or to reject it, on the basis of an imperfectly known model of the joint. The usability of the model is analysed to determine the accuracy to which the model must be verified in order for the decision to be reliable.

The mechanical model is the following relation between force $f$ and dimensionless strain $x$:

$$f(x) = kx + \sum_{n=0}^{N} \beta_n x^n \tag{10}$$

where the strain is constrained to the range $|x| \leq x_{\max}$. The uncertainty model is a spheroidal convex model for the vector $\beta = (\beta_0, \ldots, \beta_N)^T$ of polynomial coefficients:

$$\mathscr{B}(\alpha) = \{\beta\colon \beta^T\beta \leqslant \alpha^2\} \tag{11}$$

The decision will be to accept the joint if and only if the force at maximum strain deviates from the nominal force $kx_{\max}$ by no more than a critical value:

$$D(\beta) = \text{yes}, \quad \text{if and only if } |f(x_{\max}) - kx_{\max}| \leqslant \varepsilon_{\mathrm{cr}} \tag{12}$$

In other words, the non-linearity of the joint has been modeled empirically in equation (10) with the coefficient vector $\beta$ whose value may not be precise. The joint is acceptable if its performance, as predicted by the empirical model, is within $\varepsilon_{\mathrm{cr}}$ of the linear model $kx$. The decision is based on the empirical $\beta$, and this decision is robust if large error in $\beta$ does not jeopardise the correctness of the decision.

Note that the decision is 'yes' when $\alpha = 0$. So, the robust usability is the greatest value of the non-linear uncertainty parameter, $\alpha$, for which the decision is the same, namely 'yes', for all non-linear models in $\mathscr{B}(\alpha)$:

$$\hat{\alpha} = \sup\left\{\alpha\colon \left|\sum_{n=0}^{N} \beta_n x_{\max}^n\right| \leqslant \varepsilon_{\mathrm{cr}} \quad \text{for all } \beta \in \mathscr{B}(\alpha)\right\} \tag{13}$$

To evaluate the usability the maximum of the sum in this expression needs to be determined, as $\beta$ varies on $\mathscr{B}(\alpha)$, and then this maximum should be equated to $\varepsilon_{cr}$. Define the function:

$$g(x) = \sqrt{\frac{x^{2(N+1)} - 1}{x^2 - 1}} \tag{14}$$

The maximum sum is found to be $\alpha g(x_{max})$. Equating this to $\varepsilon_{cr}$ results in the robust usability:

$$\hat{\alpha} = \frac{\varepsilon_{cr}}{g(x_{max})} \tag{15}$$

When $\hat{\alpha}$ is large, the decision is robust to the non-linear model uncertainty, while if $\hat{\alpha}$ is small the decision is fragile so the model usability is low.

What values of $\hat{\alpha}$ are 'small' or 'large'? Since $x$ is dimensionless, $\beta_n$, $\alpha$ and $\hat{\alpha}$ all have units of force. One reasonable physical calibration of the magnitude of $\hat{\alpha}$ is to compare it against the maximum force of the element. This maximum force, based on the largest acceptable uncertainty model, $\mathscr{B}(\hat{\alpha})$, is $kx_{max} + \hat{\alpha}g(x_{max})$. The ratio of $\hat{\alpha}$ to this maximum force is:

$$\frac{\hat{\alpha}}{kx_{max} + \hat{\alpha}g(x_{max})} = \frac{\varepsilon_{cr}/g(x_{max})}{kx_{max} + \varepsilon_{cr}} \tag{16}$$

The robustness is 'small', 'moderate' or 'large' if this quantity is $\ll 1$, $\approx 1$ or $\gg 1$, respectively.

$g^2(x) \geqslant 1$ so equation (15) shows that the usability is less than $\varepsilon_{cr}$. In other words, in order to decide reliably if the joint is acceptable, it is necessary that the model accuracy, $\alpha$, be better than the acceptable error of the joint, $\varepsilon_{cr}$.

### 3.4. EXAMPLE 4: EXAMPLE 3 CONTINUED, $J$ SERIAL UNITS

Consider $J$ identical elements in a parallel array, each represented by equation (10) with the same $\beta$-vector. If a strain $x$ is imposed on the entire array, each element also will expand or contract by $x$. Consequently, the force at the end of the array is $Jf(x)$. Equation (11) is the uncertainty model for each of the non-linear elements. The decision is to accept the array of joints if the end force at maximal strain deviates from the nominal force by no more than $\varepsilon_{cr}$:

$$D(\beta) = \text{yes}, \quad \text{if and only if} \quad |Jf(x_{max}) - Jkx_{max}| \leqslant \varepsilon_{cr} \tag{17}$$

For any coefficient vector in $\mathscr{B}(\alpha)$, the maximum force of a single element strained to $x$ is, as before, $kx + \alpha g(x)$. So, the maximal end force, when the array is strained to $x_{max}$, is $J[kx_{max} + \alpha g(x_{max})]$. So, the robustness of the decision, with a $J$-element array, is:

$$\hat{\alpha}_J = \frac{\varepsilon_{cr}}{Jg(x_{max})} \tag{18}$$

This indicates that the robust usability decreases monotonically with the number of elements in the serial array.

### 3.5. EXAMPLE 5: EXAMPLE 3 CONTINUED, COMPARING EXPANSION FUNCTIONS

Returning to equation (10), represent the non-linear term with an expansion in functions $\phi_0(x), \ldots, \phi_N(x)$ different from power functions:

$$f(x) = kx + \sum_{n=0}^{N} \beta_n \phi_n(x) \tag{19}$$

As in example 3, the uncertainty model for the coefficient vector $\beta$ is equation (11) and the decision algorithm is equation (12). The robust usability is equation (13) with $x^n$ replaced by $\phi_n(x)$. The robust reliability is now:

$$\hat{\alpha}_\phi = \frac{\varepsilon_{cr}}{\sqrt{\sum_{n=0}^{N} \phi_n^2(x_{max})}} \tag{20}$$

Can the usability of the $\phi$-function model be compared to the polynomial model, whose robustness is henceforth denoted $\hat{\alpha}_x$ ?

This can be done very easily, if it is assumed that both the polynomial and the $\phi$-function models are sufficiently rich to reproduce the non-linear behaviour of the element accurately, for some choice of the coefficient vector $\beta$ in the convex model of size $\hat{\alpha}_x$ or $\hat{\alpha}_\phi$, respectively. The uncertainty is a parameter uncertainty: it is not known which $\beta$ reproduces the true behaviour, and the usability of the model is evaluated as the greatest residual imprecision of $\beta$ which has no effect on the decision. Given the 'model-richness' assumption, the comparison of alternative models is 'fair' since both classes of models can represent accurately the non-linear behaviour and one wishes to know which does so with the least sensitivity to uncertainty. On the other hand, if the form of the uncertain part of the model was inherently unable to capture the non-linear contribution, for example if an insufficiently large expansion order $N$, was chosen, then this would lead to a structural uncertainty. If the model class fails to include the true behaviour, insensitivity to uncertainty (a large value for $\hat{\alpha}$) would not imply good model usability. Therefore, comparing structurally uncertain models is more problematical than comparing parametrically uncertain models.

Upon the assumption that the uncertainty is only in the parameters, the relative usability of the polynomial to the $\phi$-function model is found by comparing equations (15) and (20):

$$\frac{\hat{\alpha}_x}{\hat{\alpha}_\phi} = \sqrt{\frac{\sum_{n=0}^{N} \phi_n^2(x_{max})}{\sum_{n=0}^{N} x_{max}^{2n}}} \tag{21}$$

where the denominator is just $g(x)$ before summing the geometric sequence.

## 4. USABILITY OF A MULTIDIMENSIONAL FREQUENCY RESPONSE MODEL

This section evaluates the robust usability of a multidimensional frequency response model of a mechanically vibrating structure. First, the three components of the analysis are outlined: the mathematical model of the system, the uncertainty models, and the decision algorithm.

The mathematical model is a frequency-domain relation between the Laplace transform of the input load vector $u$ and of the response vector $y$:

$$y(\omega) = \Gamma(\omega)u(\omega) \tag{22}$$

where $\Gamma$ is the modeled flexibility matrix.

Uncertainties are considered in both the input and the model. The uncertainty model for the inputs is a convex model represented as $\mathscr{U}(\alpha_u, \bar{u})$, where $\alpha_u$ is the uncertainty parameter for the input and $\bar{u}$ is the nominal input. A typical input uncertainty model is an ellipsoid-bound convex model:

$$\mathscr{U}(\alpha_u, \bar{u}) = \{u\colon (u-\bar{u})^H W(u-\bar{u}) \leqslant \alpha_u^2\} \tag{23}$$

where $W$ is a positive definite real symmetric matrix and $H$ implies hermitian matrix transposition.

The modeled flexibility matrix $\Gamma$ depends on parameters $p$ as well as on frequency, so $\Gamma(\omega, p)$ is sometimes used. Model uncertainty is expressed in terms of uncertainty in the parameters by a convex model $\mathscr{P}(\alpha_p, \bar{p})$, where $\alpha_p$ is the uncertainty parameter for the model and $\bar{p}$ is the nominal model. For instance, $\bar{p}$ might represent the model obtained by an updating procedure. A typical convex model for parameters is interval uncertainty:

$$\mathscr{P}(\alpha_p, \bar{p}) = \{p\colon |p_i - \bar{p}_i| \leqslant \alpha_p \bar{p}_i\} \tag{24}$$

The designer or analyst will need to make a plethora of different decisions under various circumstances. This section considers a situation in which one needs to determine the spatial stability of the physical system subject to uncertain input, based on output predictions of a specific mathematical model. The 'decision', $D$, is to accept or reject the system depending on whether or not the range of the model-predicted response to the uncertain loads is less or greater than an acceptable threshold, $\varepsilon_{cr}$. Formally, the decision algorithm is:

$$D(p) = \text{yes, if and only if} \quad \max_{u \in \mathscr{U}(\alpha_u, \bar{u})} \|\Gamma(\omega, p)u\| \leqslant \varepsilon_{cr} \tag{25}$$

The robust usability of the model for such a decision is: how large can the model uncertainty be and still maintain the same decision for all models in the set? Just like equation (1), the robustness of the decision is the greatest value of $\alpha_p$ such that $D(p)$ is the same for all models in $\mathscr{P}(\alpha_p, \bar{p})$:

$$\hat{\alpha} = \sup\{\alpha_p : D(p) = \text{constant}, \quad \text{for all } p \in \mathscr{P}(\alpha_p, \bar{p})\} \tag{26}$$

### 4.1. EXAMPLE 6: ZERO NOMINAL INPUT

Suppose that the nominal input is zero, so $\bar{u} = 0$. In this case, the maximization in equation (25) becomes an eigenvalue problem. The maximum normed response is:

$$\max_{u \in \mathscr{U}(\alpha_u, \bar{u})} \left\| \Gamma(\omega, p)u \right\| = \alpha_u \sqrt{\max \operatorname{eig}\,[W^{-1/2}\Gamma^H(\omega, p)\Gamma(\omega, p)W^{-1/2}]} \tag{27}$$

Let $\mu(p)$ denote the maximum eigenvalue whose square root appears in this expression.

Suppose that the decision is 'yes' and the system is acceptable with the nominal model, $\bar{p}$. That is, $\alpha_u \sqrt{\mu(\bar{p})} \leqslant \varepsilon_{cr}$. According to equation (26), the robustness of the decision is the greatest value of the model uncertainty parameter $\alpha_p$ for which the decision is 'yes' for all models in $\mathscr{P}(\alpha_p, \bar{p})$:

$$\hat{\alpha} = \sup\{\alpha_p : \alpha_u \sqrt{\mu(p)} \leqslant \varepsilon_{cr} \quad \text{for all } p \in \mathscr{P}(\alpha_p, \bar{p})\} \tag{28}$$

From equations (24) and (28) it can be seen that the robust usability index, $\hat{\alpha}$, like the model uncertainty parameter $\alpha_p$, is dimensionless and has the meaning of a fractional error in the model parameters. Suppose it is known that a plausible value for the fractional error of the model parameter is $f$. The model is very robust, moderately so, or not at all robust as $\alpha \gg f$, $\hat{\alpha} \approx f$ of $\hat{\alpha} \ll f$, respectively.

### 4.2. EXAMPLE 7: NON-ZERO NOMINAL INPUT

Consider the more complex but realistic situation in which the nominal input, $\bar{u}$, is different from zero. For notational convenience define $V = \Gamma^H(\omega, p)\Gamma(\omega, p)$. Now one finds:

$$\max_{u \in \mathscr{U}(\alpha_u, \bar{u})} \left\| \Gamma(\omega, p)u \right\| = \sqrt{(v + \bar{u})^H V (v + \bar{u})} \tag{29}$$

where $v$ is the solution of:

$$[V - \lambda W]v = -V\bar{u} \tag{30}$$

and the unknown multiplier $\lambda$ is chosen so that $v$ satisfies:

$$v^T W v = \alpha_u^2 \tag{31}$$

Equations (29)–(31) enable one to implement the decision algorithm for any choice of the parameter vector $p$ by using relation (25). The robustness is the greatest value of $\alpha_p$ such that the decision is the same for all models in $\mathscr{P}(\alpha_p, \bar{u})$, as expressed in equation (26).

### 4.3. EXAMPLE 8: COMPARING TEST AND REFERENCE MODELS

In some situations a test model $\Gamma^t$ has been developed and the engineering decision to be made is whether to accept or reject the test model on the basis of its fidelity to an accepted reference model, $\Gamma^r$. The input uncertainty is represented by the convex model of equation (23) and the uncertainty in the test model is expressed by equation (24). Instead of equation (25), the decision algorithm is:

$$D(p) = \text{yes}, \quad \text{if and only if} \quad \max_{u \in \mathscr{U}(\alpha_u, \bar{u})} \| [\Gamma^t(\omega, p) - \Gamma^r(\omega, p)]u \| \leqslant \varepsilon_{\text{cr}} \tag{32}$$

The robust usability of this decision is evaluated from equation (26).

In this example, the test model is a specific realisation, a given numerical flexibility matrix, denoted $\Gamma^t$. The decision algorithm, equation (32), can be implemented for this $\Gamma^t$ and results in a specific decision: either accept or reject the test model. Suppose one decides to accept $\Gamma^t$. Is this decision robust or fragile to uncertainty in $\Gamma^t$? It might be that if $\Gamma^t$ were only slightly different it would fail the test in equation (32) and it would be rejected. In this case, the decision to accept $\Gamma^t$ is fragile and should possibly be reversed or at least re-evaluated. Or, perhaps even models quite different from the specific realization, $\Gamma^t$, would also be accepted, indicating that the decision is robust and reliable.

### 4.4. EXAMPLE 9: USABILITY FOR MULTIPLE DECISIONS

Engineering decisions often comprise several component decisions. For example, the structure may be accepted for use if the maximum deflection in response to uncertain loads is less than a critical value, based on a test model $\Gamma^t$, and if this test model has sufficient fidelity to a reference model $\Gamma^r$. In other words, the decision to accept or reject the structure

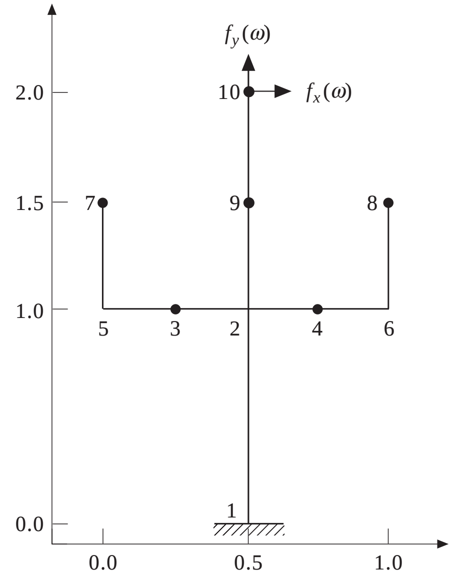


Figure 2. Geometry of the model FORK. Numbers indicate nodal points; ●, lumped mass; →, excitation.

is a combination of the decisions in equation (25) and (32). Formally, this composite decision algorithm is expressed:

$$D(p) = \text{yes} \quad \text{if and only if} \quad D_1(p) = \text{yes} \quad \text{and} \quad D_2(p) = \text{yes} \tag{33}$$

where

$$D_1(p) = \text{yes}, \quad \text{if and only if} \quad \max_{u \in \mathscr{U}(\alpha_u, \bar{u})} \| \Gamma^t(\omega, p) u \| \leqslant \varepsilon_{\text{cr},1} \tag{34}$$

$$D_2(p) = \text{yes}, \quad \text{if and only if} \quad \max_{u \in \mathscr{U}(\alpha_u, \bar{u})} \| [\Gamma^t(\omega, p) - \Gamma^r(\omega, p)] u \| \leqslant \varepsilon_{\text{cr},2} \tag{35}$$

The robust usability is evaluated from equation (26). This binary decision can be generalised to include an arbitrary number of component decisions connected by logical operations such as ‘and’, ‘or’ and ‘nor’. One can also construct a more complex hierarchy of sub-decisions with conditional branches and linguistic qualifiers. In all cases, the usability of the model is the robustness of the final decision, $D$.

## 5. ILLUSTRATIVE NUMERICAL EXAMPLE

This section considers a numerical implementation of example 8. The structure FORK is a bidimensional clamped-free beam structure, composed of nine Euler beam elements, six lumped masses, comprising a total of 27 degrees of freedom. The general geometrical characteristics of the test case are shown in Fig. 2 and detailed description of the nominal model can be found in [6].

In the context of the present example, the initial FORK model is considered as a reference model $\Gamma^r$ and the robust usability $\hat{\alpha}$ of this model is evaluated with respect to parameter uncertainty of a test model $\Gamma^t$, where the loads are also uncertain.

The loads are defined by the non-zero components of the input vector indicated by the two arrows at node 10 in Fig. 2. The input uncertainty model is defined by equation (23) with $W$ as the identity matrix and $\bar{u} = 0$. The parameter space of the test model, $\Gamma^t$, is characterised by a single uncertain scalar coefficient $p$ representing the simultaneous modification of the quadratic moments of inertia of all nine beams, that is to say:

$$\mathscr{I} = p\overline{\mathscr{I}} \in \Re^{9,1} \tag{36}$$

The interval uncertainty in $p$ is represented by the convex model $\mathscr{P}(\alpha_p)$ defined by:

$$\mathscr{P}(\alpha_p) = \{p: |p-1| \leqslant \alpha_p\} \tag{37}$$

The decision algorithm is specified in equation (32), and the robust reliability with respect to model uncertainty is defined in equation (26).

The maximum which is evaluated by the decision algorithm in equation (32) is the largest discrepancy between the reference and test models, for all inputs in the load–uncertainty set $\mathscr{U}(\alpha_u)$. The solution of this maximisation problem leads to a symmetric eigenvalue problem of the form:

$$[\Delta\Gamma^T(\omega, p)\Delta\Gamma(\omega, p) - \varepsilon_v\, I]u_v = 0 \tag{38}$$

whose largest eigenvalue corresponds to the maximal absolute response error between the test model, defined by the specific value of the coefficient $p$, and the reference model. Let $\Delta\Gamma = \Gamma^t - \Gamma^r$, and let $\varepsilon$ denote the maximum eigenvalue. This optimisation problem is illustrated by Fig. 3 where the maximum error is plotted as a function of frequency for $p = 0.9$ and $\alpha_u = 1$.

The evaluation of the robustness with respect to model uncertainty involves finding the supremum defined in equation (26): the greatest value of the model-error uncertainty parameter $\alpha_p$ for which the decision is constant. This optimisation problem is illustrated at a frequency $f_0 = 36$ Hz in Fig. 4. In the present case, the intersection between the maximal absolute error curve and the critical level $\varepsilon_{cr}$ gives a robust model usability of $\hat{\alpha}_p = 0.3$. That is, the decision is the same for all test models within the model-uncertainty set $\mathscr{P}(\alpha_p)$ for any $\alpha_p \leqslant 0.3$. In other words, the decision is only moderately robust with respect to uncertainty in the modeled moments of inertia, since a modeling error of more than 30% in the moments of inertia is liable to cause a change in the decision.

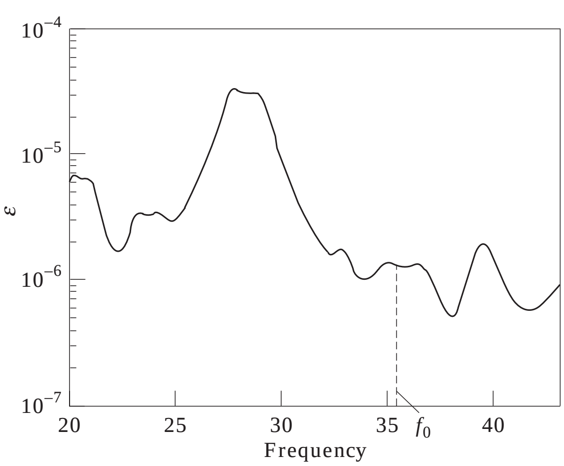


Figure 3. Maximum error ε as a function of frequency for $p = 0.9$.

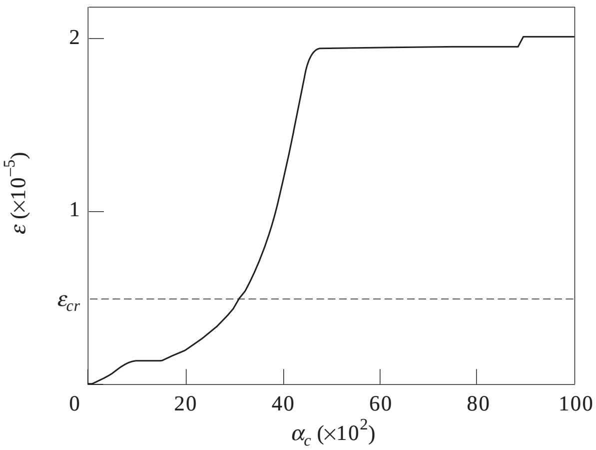


Figure 4. Maximum error $\varepsilon$ as a function of $\alpha_p$.

## 6. CONCLUSION

Mathematical models are tools for making rational decisions in the design and analysis of mechanical systems. Not only must a model be verified and validated against empirical data, but it must also be evaluated with respect to its end use: the decisions which rely upon the model. The measure of model usability developed here is: a model is usable to the extent that decisions based on the model are robust to the associated uncertainties. This can be implemented precisely for a wide spectrum of specific mechanical decision processes, as has been illustrated in the examples. Convex models have been used to represent uncertainties both in the operational environment and in the system itself as represented by the model. A methodology has been developed for evaluating model usability in terms of engineering decisions, and its application has been demonstrated to both linear and non-linear mechanical systems. Different approaches to calibrating or interpreting the model usability have been discussed in subjective terms which are meaningful for practical engineering judgements.

Decision robustness is a reasonable measure of model usability for two reasons. First, large decision robustness means that the same decision is obtained over a wide range of realisations. Decision robustness enhances the analyst's immunity or insensitivity to unavoidable uncertainties. Robust decisions are stable with respect to unknown variations. The second advantage is more subtle, and involves the correctness, not just the stability, of the decision. If the decision is robust to model uncertainties and if a 'correct' model lies somewhere in the domain of decision robustness, then stability of the decision implies correctness. This was discussed in Section 2 following equation (1) and again explicitly in connection with example 5.

In many of the discussed examples, the model usability was evaluated with respect to model uncertainties, while input uncertainties may also have existed. From this, the analysis of model usability can be incorporated into model up-dating in the following way. The robust usability with respect to model uncertainty is evaluated from equation (26), which shows that the robustness is a function of the 'centre point' of the model, $\bar{p}$, so one writes $\hat{\alpha}(\bar{p})$. Model updating is often an iterative procedure in which one evaluates a sequence of models according to some performance criterion whose extremum selects viable models. Often a range of models, $p_1, p_2, \ldots$, are all nearly equally viable. In this case, the selection among these models can be done on the basis of their usabilities, $\hat{\alpha}(p_1)$, $\hat{\alpha}(p_2), \ldots$; the model with greatest robustness is selected. But the integration of usability-analysis in model up-dating can be even more intimate. In iterative model updating, one frequently uses some criterion such as a gradient of the performance criterion for proceeding from one iteration to the next. Since one wishes to obtain a model which has both good performance and high robustness, the gradient of the robustness may also be 'folded into' the iteration procedure.

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