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Robust Rationality and Decisions Under Severe Uncertainty

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Abstract

This paper develops a prescriptive approach to decision-making with severely uncertain information, and explores risk-taking behavior, based on non-probabilistic set-models of information-gap uncertainty. Info-gap models are well suited for representing uncertainty arising from severe lack of information, and lead naturally to a decision strategy which maximizes the decision-maker's immunity to uncertainty, while also achieving no less than a specified minimum reward. We prove a "gambler's theorem" which quantifies the trade-off between reward and immunity-to-uncertainty. This trade-off forces the decision-maker to gamble, but without employing a probabilistic framework. We present a complementary theorem expressing the trade-off between immunity and windfall reward, and a further result characterizing the antagonism between robustness to failure and opportunity for success. Next we develop a measure of risk-sensitivity based on the idea of immunity-to-uncertainty, without any probabilistic underpinning and without the assumptions of von Neumann-Morgenstern utility theory. We prove a theorem which establishes the relation between a decision maker's aversion to uncertainty and the information which is available to him. Our final theorem establishes conditions in

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which the magnitude of the decision maker's commitment of resources will increase with his fondness for risk.

1 Introduction

The recognition of individual and organizational aversion to uncertainty underlies much of the analysis of decision-making with incomplete information. In this paper we develop a prescriptive approach to decision-making based on non-probabilistic set-models of uncertainty, whose axiomatic basis and implications for inference have been studied in a previous paper [7]. Set-models of uncertainty, and in particular convex information-gap models, have developed in recent decades for quantifying severe uncertainty in a range of engineering disciplines. Info-gap models of uncertainty also match a perception of uncertainty which is widespread in management and economics. The concept of robust rationality developed here derives naturally from info-gap models, and generates a decision algorithm which is particularly suited to situations severely lacking in information. In this paper we study the risk-taking behavior of a decision maker whose inferences are based on info-gap models of uncertainty.

The pedigree of probability in theories for decision under uncertainty is impeccable, but its limitations are equally widely recognized. Not all decision makers obey probabilistic expected utility theory [20]. Not all decision makers are able or willing to perform, either explicitly or implicitly, probabilistic computations. Not all decision-making environments are informationally rich enough to support probabilistic analysis.

Robustness to environmental indeterminacy underlies the decision methodology developed here: specified goals are guaranteed (when possible) by the action selected, while at the same time the decision-maker's immunity to uncertainty is maximized. In this sense, the methodology developed here is a specific implementation of Herbert Simon's concept of "satisficing". The decision methodology of robust rationality which is proposed here is formulated in section 3. Two basic functions are introduced: the robustness function $\widehat{\alpha}(q, r_{\rm c})$ expresses the greatest info-gap uncertainty consistent with reward no less than $r_{\rm c}$ if action q is adopted. The opportunity function $\widehat{\beta}(q, r_{\rm w})$ expresses the least uncertainty consistent with reward possibly as great as $r_{\rm w}$ if action q is adopted.

Info-gap models of uncertainty, which we will describe briefly in section 2, match Simon's thought in another way as well. An info-gap model is a family of nested sets $\mathcal{C}(\alpha)$, $\alpha \geq 0$, in which the "uncertainty parameter" α determines the order of nesting as well as the degree of information-gap uncertainty. Simon writes that "In mathematical models incorporating rational and non-rational aspects of behavior, the non-rational aspects can be imbedded in the model as the limiting conditions that "bound" the area of rational adjustment." [28, p.215]. The info-gap model is one possible quantification of Simon's "bounded rationality".

In section 4 we present a "gambler's theorem" which quantifies the trade-off between reward and immunity-to-uncertainty and which is based on the robustness function $\widehat{\alpha}(q, r_{\rm c})$. This trade-off forces the decision-maker to gamble, but without requiring a probabilistic framework. This theorem epitomizes the method of robust rationality, which is based on non-probabilistic info-gap models of uncertainty. We also present a complementary theorem expressing the trade-off between immunity and windfall reward, employing the opportunity function $\widehat{\beta}(q, r_{\rm w})$.

In section 5 we study the antagonism between immunity and opportunity: the more the decision maker enhances his robustness to failure, the more he reduces his opportunity for unanticipated success, and vice versa. Surprisingly, this antagonism is not universal, and we examine a necessary and sufficient condition for it to occur.

In section 6 we explore a measure of risk-sensitivity based on the idea of immunity-to-uncertainty as expressed by the robustness function and non-probabilistic info-gap models of uncertainty. The concept of risk developed here is different from the classical Arrow-Pratt probabilistic theory. Our results are obtained without any probabilistic underpinning, and without the assumptions of von

Neumann-Morgenstern utility theory [24]: uncertainty, yes; probability, no. From the results of section 4 we view the robustness function as expressing the trade-off between certainty and reward. The robustness function $\widehat{\alpha}(q, r_{\rm c})$ expresses not only the decision maker's prior preferences, but also the extent of his ignorance of the environment and his willingness to forgo certainty in his search for obtaining reward. $\widehat{\alpha}(q, r_{\rm c})$ expresses the decision maker's attitude to the range of risky options, and characterizes the wariness or boldness of his choices. Theorem 3 and its corollaries deal with the relation between the decision maker's aversion to risk and the information which is available to him, and give quantitative expression to the following assertions about this measure of sensitivity to uncertainty: (1) risk aversion increases with info-gap uncertainty, (2) risk aversion decreases with reward, and (3) uncertainty is offset by reward. Theorem 4 and its corollaries establish conditions in which the magnitude of the decision maker's commitment of resources will increase with his fondness for risk.

2 Information-Gap Models of Uncertainty

Our quantification of uncertainty is based on non-probabilistic information-gap models. An info-gap model is a family of nested sets. Each set corresponds to a particular degree of uncertainty, according to its level of nesting. Each element in a set represents a possible realization of the uncertain event. Info-gap models, and especially convex-set models of uncertainty, have been described elsewhere, both technically [3], non-technically [2] and axiomatically [5, 7]. Specific examples of the formulation of info-gap models in a range of mechanical applications are found in [3], for seismic safety of structures in [9], for environmental problems in [19], and for project management in [10].

Uncertain quantities are vectors or vector functions. Uncertainty is expressed at two levels by info-gap models. For fixed scalar α the set $\mathcal{C}(\alpha, \widetilde{v})$ represents a degree of uncertain variability of the uncertain quantity v around the centerpoint $\widetilde{v}$. The greater the value of α, the greater the range of possible variation, so α is called the *uncertainty parameter* and expresses the information gap between what is known ($\widetilde{v}$ and the structure of the sets) and what needs to be known for an ideal solution (the exact value of v). The value of α is usually unknown, which constitutes the second level of uncertainty: the horizon of uncertain variation is unbounded.

Let $\Re$ denote the non-negative real numbers and let S be a Banach space in which the uncertain quantities v are defined. An info-gap model $\mathcal{C}(\alpha, \widetilde{v})$ is a map from $\Re \times S$ into the power set of S.

Info-gap models obey four axioms. *Nesting:* $\mathcal{C}(\alpha, \widetilde{v}) \subseteq \mathcal{C}(\alpha', \widetilde{v})$ if $\alpha \le \alpha'$. *Contraction:* $\mathcal{C}(0, 0)$ is the singleton set $\{0\}$. *Translation:* $\mathcal{C}(\alpha, \widetilde{v})$ is obtained by shifting $\mathcal{C}(\alpha, 0)$ from the origin to $\widetilde{v}$. *Linear expansion:* info-gap models centered at the origin expand linearly: $\mathcal{C}(\beta, 0) = \frac{\beta}{\alpha}\mathcal{C}(\alpha, 0)$ for all α, $\beta > 0$.

Even though info-gap models of uncertainty emerged in engineering — control theory [27], seismic design of structures [12, 13], nuclear measurements [1], mechanical analysis [8] and mechanical reliability [3] — they also match an intuition of uncertainty which is prevalent among economists and management theorists. We will mention just a few examples. March analyzes individual and organizational decision making and writes that "Uncertainty is a limitation on understanding and intelligence." [23, p.178]. Mack writes that "Uncertainty is the complement of knowledge. It is the gap between what is known and what needs to be known to make correct decisions." [22, p.1]. Galbraith, in discussing the design of complex industrial organizations, defines uncertainty as an information gap, as "the difference between the amount of information required to perform the task and the amount of information already possessed by the organization." [16, p.5]. Laufer identifies nine principles for managing projects in an era of uncertainty. He writes that one central principle in project management is that, "in dynamic conditions, influencing the future is not just making decisions early. It is more about the ability to *reduce uncertainty* and to minimize the impact of surprises." (emphasis in the original) [21, p.74]. Y.B. Choi writes that "Uncertainty is a state of doubt. It describes an indeterminate relationship between us and the environment we face. ... The very existence of uncertainty presupposes the need for judgement and decision making that will resolve this state." [11, p.27].

We will encounter several examples of info-gap models of uncertainty in this paper. Other instances are found in the references.

3 Robust Rationality

We begin with a generic formulation of robust rationality, which we will illustrate in the next section and in several subsequent examples. Let the vector q represent the decision-maker's actions, which incur a consequence depending upon q and upon an uncertain vector or vector function v belonging to an info-gap model $\mathcal{C}(\alpha, \widetilde{v})$. The decision maker knows his scalar reward function, $R(q, v)$, which he could use, together with $\widetilde{v}$, to maximize his nominal reward. The reward function $R(q, v)$ must be construed very broadly, and can represent a performance requirement in vibrational analysis or other engineering design problems [4], an economic or managerial goal [10], or any other scalar objective. R need not be a von Neumann-Morgenstern utility function. The classical full-information strategy is to choose an optimal action $\widetilde{q}$ as:

$$R(\widetilde{q}, \widetilde{v}) = \max_{q \in \mathcal{Q}} R(q, \widetilde{v}) \tag{1}$$

where $\mathcal{Q}$ is the set of available actions.

Now consider the robust strategy. The robustness is the greatest value of the uncertainty parameter for which the reward is no less than a least acceptable or "critical" value, r_{c}. Define the set of α-values for which the critical reward is guaranteed:

$$\mathcal{A}(q, r_{\mathrm{c}}) = \left\{ \alpha : \min_{v \in \mathcal{C}(\alpha, \widetilde{v})} R(q, v) \ \geq \ r_{\mathrm{c}} \right\} \tag{2}$$

The robustness function can now be expressed as:

$$\widehat{\alpha}(q, r_{\mathrm{c}}) = \max_{\alpha \in \mathcal{A}(q, r_{\mathrm{c}})} \alpha \tag{3}$$

If the set $\mathcal{A}(q, r_{\mathrm{c}})$ is empty then the reward r_{c} cannot be achieved at any level of uncertainty and we define $\widehat{\alpha}(q, r_{\mathrm{c}}) = 0$. When $\widehat{\alpha}(q, r_{\mathrm{c}})$ is large then the decision maker is immune to a wide range of variation, while if $\widehat{\alpha}(q, r_{\mathrm{c}})$ is small then even small fluctuations can lead to reward less than the critical value r_{c}. Thus "bigger is better" for the robustness $\widehat{\alpha}$.

The decision-maker's optimal robust strategy is to choose an action $\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})$ which maximizes the robustness:

$$\widehat{\alpha}(\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}}), r_{\mathrm{c}}) = \max_{q \in \mathcal{Q}} \widehat{\alpha}(q, r_{\mathrm{c}}) \tag{4}$$

The logical structure of this decision algorithm as a "severe test" is discussed further in [7].

Robustness to uncertainty is important when we concentrate on the pernicious possibilities entailed by unknown variation. However, variations can be propitious and surprises can be beneficient. The dynamic flexibility which is so important for survival, as Rosenhead had stressed [26], thrives on emergent opportunities. We can study this aspect of decisions based on robust rationality with the aid of a function which is the logical complement of the robustness function $\widehat{\alpha}$.

The "opportunity function" $\widehat{\beta}$ is constructed in a manner similar to $\widehat{\alpha}$. Let r_{w} represent the decision maker's wildest dreams of reward: a value of reward much greater than the critical reward r_{c} needed for bare survival. Naturally, dreams of success can be more or less wild, so r_{w} is a variable just as r_{c} is a variable assessment of the decision maker's minimum demands. Define the set of values of the uncertainty parameter consistent with maximum reward no less than r_{w}:

$$\mathcal{B}(q, r_{\mathrm{w}}) = \left\{ \alpha : \max_{v \in \mathcal{C}(\alpha, \widetilde{v})} R(q, v) \ \geq \ r_{\mathrm{w}} \right\} \tag{5}$$

$\mathcal{B}(q, r_w)$ is the analog of $\mathcal{A}(q, r_c)$ in eq.(2). The uncertain future is propitious if even small variation can lead to unhoped-for success. The opportunity function is defined as the least uncertainty consistent with maximum reward no less than r_w:

$$\widehat{\beta}(q, r_w) = \min_{\alpha \in \mathcal{B}(q, r_w)} \alpha \tag{6}$$

If the set $\mathcal{B}(q, r_w)$ is empty then the windfall reward r_w cannot be achieved at any level of uncertainty and we define $\widehat{\beta}(q, r_w) = \infty$. If $\widehat{\beta}(q, r_w)$ is small, then small deviation from nominal or anticipated conditions can lead to reward as large as r_w. Uncertainty is propitious if even small fluctuations can be favorable. On the other hand, if $\widehat{\beta}(q, r_w)$ is large, then reward as great as r_w can occur only at large deviation from the norm. For any value of the decision maker's highest aspiration, r_w, a small value of $\widehat{\beta}(q, r_w)$ is preferable over a large value. Thus "big is bad" and "small is good" with the opportunity function $\widehat{\beta}(q, r_w)$, exactly the reverse of the calibration of the robustness function $\widehat{\alpha}(q, r_c)$. $\widehat{\beta}(q, r_w)$ is the complement of the robustness function $\widehat{\alpha}(q, r_c)$, defined in eq.(3). Both $\widehat{\alpha}$ and $\widehat{\beta}$ are immunities: $\widehat{\alpha}$ is the immunity to intolerable failure while $\widehat{\beta}$ is the immunity to sweeping success. While $\mathcal{A}(q, r_c)$ is bounded above by $\widehat{\alpha}(q, r_c)$, we see that $\mathcal{B}(q, r_w)$ is bounded below by $\widehat{\beta}(q, r_w)$.

Instead of maximizing the immunity to failure, as in eq.(4), the decision maker may choose a strategy which minimizes the immunity to opportunity, in which case the "optimal opportunity" strategy is $\widehat{q}_w(r_w)$ satisfying:

$$\widehat{\beta}(\widehat{q}_w(r_w), r_w) = \min_{q \in \mathcal{Q}} \widehat{\beta}(q, r_w) \tag{7}$$

where $\mathcal{Q}$ is the set of available actions.

4 Robustness vs. Reward: Gambler's Theorems

The first theorem in this section expresses the trade-off between the minimum reward demanded by the decision-maker, r_c, and the robustness against uncertainty $\widehat{\alpha}(q, r_c)$ which he can attain with action-vector q. While this trade-off is not surprising, the theorem illuminates the type of gambling involved in this decision procedure, even though no probabilistic information in employed. Also, it shows how the decision maker may be led to modify his choice of r_c as a result of the analysis. The second theorem is complementary to the first, and expresses the trade-off between immunity and the attainment of unanticipated windfall reward based on the opportunity function $\widehat{\beta}(q, r_w)$.

We now state what, in the logic of robust rationality, could be called a gambler's theorem.

Theorem 1 *Let $\mathcal{C}(\alpha, \widetilde{v})$ be an info-gap model, let $R(q, v)$ be a uniformly continuous reward function, and let r_1 and r_2 be two values of reward where $r_1 < r_2$. If $\widehat{\alpha}(q, r_1)$ and $\widehat{\alpha}(q, r_2)$ exist and are positive for $\mathcal{C}(\alpha, \widetilde{v})$, then:*

$$\widehat{\alpha}(q, r_1) > \widehat{\alpha}(q, r_2) \tag{8}$$

That is, the robustness function is strictly monotonically decreasing on the set of reachable rewards. Proofs are presented in the Appendix. Corollary m/n will denote the mth corollary of theorem n.

Corollary 1/1 *Let $\mathcal{C}(\alpha, \widetilde{v})$ be an info-gap model, let $R(q, v)$ be a uniformly continuous reward function, let r_1 and r_2 be two reward values where $r_1 < r_2$, and let $\widehat{q}_c(r_1)$ and $\widehat{q}_c(r_2)$ be the corresponding maximally robust strategies. Then:*

$$\widehat{\alpha}(\widehat{q}_c(r_1), r_1) > \widehat{\alpha}(\widehat{q}_c(r_2), r_2) \tag{9}$$

That is, the maximal robustness is strictly monotonically decreasing on the set of reachable rewards.

Theorem 1 and its corollary show the inexorable trade-off between reward and immunity-to-uncertainty. The decision-maker must choose his position on the spectrum of this trade-off. This theorem also shows that the decision-maker need not make an irrevocable prior choice of the critical reward $r_{\rm c}$. Rather, he can adopt $r_{\rm c}$ in light of the analysis of his robustness to ambient uncertainties. For instance, he may find that he can demand greater reward without substantially increasing his vulnerability to uncertainty. This (very pleasing) outcome of the analysis may well induce him to demand greater profit. Alternatively, the decision maker may find that by slightly diminishing his demanded reward he is able to greatly enhance his immunity to uncertainty. Again, the analysis may lead the decision maker to modify his preference. In short, the robustness $\widehat{\alpha}(q, r_{\rm c})$, viewed as a function of critical reward, is a decision-support tool with which the decision-maker assesses his options and their consequences. We illustrate this with the next example.

Example 1 Envelope-bound info-gap model. Consider a static production situation in which the manufacturer must choose the quantity q which he will produce, and let us assume this to equal the quantity sold. Let $p(q)$ be the price per item, which depends on the quantity produced, and let $c(q)$ be the total cost of producing q items. The reward function is the profit: $R(q) = p(q)\,q - c(q)$.

Suppose that the demand function $p(q)$ is known exactly, but that the cost function $c(q)$ varies in an unknown manner within an envelope of unknown size. The nominal production-cost function $\widetilde{c}(q)$ is known, and the shape of the envelope of uncertain variation of the actual cost function changes in a known way with q. The envelope-bound info-gap model is suitable to describe this incomplete prior information:

$$\mathcal{C}_{\rm e}(\alpha, \widetilde{c}) = \left\{ c(q) : \ |c(q) - \widetilde{c}(q)| \le \alpha \psi(q) \right\}, \quad \alpha \ge 0 \tag{10}$$

$\widetilde{c}(q)$ and $\psi(q)$ are known, and we have a family of nested convex sets of possible production-cost functions. $\psi(q)$ determines the *shape* of the envelope within which $c(q)$ varies in an unknown way, while the uncertainty parameter α determines the *size* of the envelope. Employing eqs.(2) and (3), the robustness of production-volume q becomes:

$$\widehat{\alpha}_{\rm e}(q, r_{\rm c}) = \frac{p(q)\,q - r_{\rm c} - \widetilde{c}(q)}{\psi(q)} \tag{11}$$

As a specific case let the envelope function in the info-gap model of eq.(10) be linear: $\psi(q) = q$. Let the nominal total cost function represent dis-economies of scale at large production volume in the following way: $\widetilde{c}(q) = c_1 q - c_2 q^{\xi}$ where $0 < \xi < 1$ and the c_n's are such that the nominal cost $\widetilde{c}(q)$ is positive. Since c_2 is positive and $0 < \xi < 1$ we find that the cost per item increases with production volume. Consider pure competition, so the demand function which the firm sees is constant: $p(q) = p_o$, where $p_o > c_1$. Eq.(11) is the robustness to uncertainty at demanded reward $r_{\rm c}$ and production volume q. The optimal volume is obtained by maximizing $\widehat{\alpha}(q, r_{\rm c})$ on q, as in eq.(4), with the following results:

$$\widehat{q}_{\rm c}(r_{\rm c}) = \left(\frac{r_{\rm c}}{(1-\xi) c_2} \right)^{1/\xi}, \quad \widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm c}) = (p_o - c_1) \left(1 + \gamma {r_{\rm c}}^{(\xi - 1)/\xi} \right) \tag{12}$$

where γ is a constant depending on ξ, p_o, c_1 and c_2 but not on $r_{\rm c}$. $\widehat{q}_{\rm c}(r_{\rm c})$ is the optimal decision for critical reward $r_{\rm c}$, and $\widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm c})$ is the corresponding maximal robustness.

The optimal robustness is shown in fig. 1 for $\xi = \gamma = 0.5$. One sees that, for large values of demanded reward, the robustness decreases slowly, suggesting that if the decision maker is willing to accept these low levels of immunity then he might as well "go to the limit" and choose $\widehat{q}$ corresponding to a very large $r_{\rm c}$. On the other hand, the figure also shows rapidly increasing robustness at low values of $r_{\rm c}$, implying that, if modest reward is acceptable, then the immunity can be substantially enhanced by demanding small return. ■

Figure 1: $\widehat{\alpha}(\widehat{q}, r_c)$ versus r_c, for example 1.

The following theorem and its corollary are the analogs of theorem 1 and corollary 1/1.

Theorem 2 *Let $\mathcal{C}(\alpha, \widetilde{v})$ be an info-gap model, let $R(q, v)$ be a uniformly continuous reward function, and let r_1 and r_2 be two values of reward where $r_1 < r_2$. If $\widehat{\beta}(q, r_1)$ and $\widehat{\beta}(q, r_2)$ exist and are positive for $\mathcal{C}(\alpha, \widetilde{v})$, then:*

$$\widehat{\beta}(q, r_1) < \widehat{\beta}(q, r_2) \tag{13}$$

That is, the opportunity function is strictly monotonically increasing on the set of reachable rewards.

Corollary 1/2 *Let $\mathcal{C}(\alpha, \widetilde{v})$ be an info-gap model, let $R(q, v)$ be a uniformly continuous reward function, let r_1 and r_2 be two reward values where $r_1 < r_2$, and let $\widehat{q}_w(r_1)$ and $\widehat{q}_w(r_2)$ be the corresponding optimal opportunity strategies. Then:*

$$\widehat{\beta}(\widehat{q}_w(r_1), r_1) < \widehat{\beta}(\widehat{q}_w(r_2), r_2) \tag{14}$$

That is, the optimal opportunity function is strictly monotonically increasing on the set of reachable rewards.

Example 2 Immunity to success and to failure. Let us continue example 1 and demonstrate the use of the opportunity function $\widehat{\beta}(q, r_w)$ as a decision tool supplementary to the robustness function $\widehat{\alpha}(q, r_c)$.

$\widehat{q}_c(r_c)$ denotes the action which maximizes the robustness, based on demanded critical reward r_c, given in eq.(12). Fig. 1 is a plot of $\widehat{\alpha}(\widehat{q}_c(r_c), r_c)$ versus r_c. In discussing fig. 1 we showed how the decision maker may be induced to modify his preference for reward based on the varying steepness of the $\widehat{\alpha}$-curve. How does the opportunity function $\widehat{\beta}(q, r_w)$ vary over the same range of actions q while r_w is fixed? How might this influence the decision maker's preferences?

$\widehat{\beta}(q, r_w)$ is a function of the decision maker's "wildest dream" of reward r_w and of his action q. We can evaluate $\widehat{\beta}(q, r_w)$ at the action $\widehat{q}_c(r_c)$ which optimizes the immunity to failure. One finds:

$$\widehat{\beta}(\widehat{q}_c(r_c), r_w) = (p_o - c_1)\left[-1 + \frac{1-\xi}{\xi}\left(\frac{r_w}{r_c} - \frac{1}{1-\xi}\right)\gamma {r_c}^{(\xi-1)/\xi}\right] \tag{15}$$

where ξ and γ are defined as in eq.(12). This is the immunity to success as a function of the decision maker's action which maximizes his immunity to failure. Recall that "big is bad" for $\widehat{\beta}$ while the reverse is true for $\widehat{\alpha}$.

Figure 2: $\widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm c})$ (solid) and $\widehat{\beta}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm w})$ (dash) versus $r_{\rm c}$. For example 2.

The solid line in fig. 2 shows the same optimal robustness curve as in fig. 1, together with the opportunity function of eq.(15) (dashed line). As before, $\xi = \gamma = 0.5$ and $r_{\rm c}$ varies from 0.1 to 1, while the value of $r_{\rm w}$ is 2, implying that the ratio $r_{\rm w}/r_{\rm c}$ varies from 2 to 20.

At large values of $r_{\rm c}$ we find that the value of $\widehat{\beta}$ is desirable (low) but $\widehat{\alpha}$ is undesirable (also low). In other words, when the immunity to failure is poor we find satisfyingly large opportunity of success. However, at small values of $r_{\rm c}$ we find $\widehat{\beta}$ increasing precipitously: by increasing the immunity to failure we also increase the immunity to unhoped-for reward. In the low range of $r_{\rm c}$-values we find that $\widehat{\beta}$ may restrain the decision maker's inclination to enhance his robustness against failure since by doing so he also relinquishes opportunities for sweeping success. In the high range of $r_{\rm c}$ values, $\widehat{\alpha}$ restrains the decision maker since his vulnerability to failure is increasing. Thus $\widehat{\alpha}$ and $\widehat{\beta}$ are antagonistic to one another and "pull" the decision maker in opposite directions. The decision maker may use both functions to explore and contrast his preferences for immunity to failure and opportunity for success. ■

5 Antagonistic Immunities

In this section we study some theoretical aspects of the phenomenon of "antagonistic immunities" illustrated in example 2 and in fig. 2.

Definition 1 Antagonistic immunities. *Robustness and opportunity functions $\widehat{\alpha}(q, r_{\rm c})$ and $\widehat{\beta}(q, r_{\rm w})$ are* antagonistic *at rewards $r_{\rm c}$ and $r_{\rm w}$ if:*

$$\frac{\partial \widehat{\beta}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm w})}{\partial r_{\rm c}} \; < \; 0 \tag{16}$$

That is, under 'antagonistic immunities' the opportunity function decreases with increasing demanded no-failure reward $r_{\rm c}$, when evaluated at the optimal action for immunity to failure and at fixed wildest-dream reward $r_{\rm w}$. What is 'antagonistic' is that the more the decision maker enhances his robustness, the more he reduces his opportunity for unplanned success, and vice versa.

Robustness to failure is not invariably antagonistic to opportunity for success. In order to investigate this we define two functions:

$$\mu_1(\alpha, q) = \min_{v \in \mathcal{C}(\alpha, \widetilde{v})} R(q, v), \qquad \mu_2(\alpha, q) = \max_{v \in \mathcal{C}(\alpha, \widetilde{v})} R(q, v) \tag{17}$$

The following properties are readily verified for these functions:

1. $\mu_1(0,q) = \mu_2(0,q) = R(q,\widetilde{v})$, because $\mathcal{C}(0,\widetilde{v})$ is the singleton set $\{\widetilde{v}\}$.

2. $\mu_1(\alpha,q)$ decreases monotonically in α and the robustness function $\widehat{\alpha}(q,r)$ is the solution (for α) of:

$$\mu_1(\alpha,q) = r \tag{18}$$

3. $\mu_2(\alpha,q)$ increases monotonically in α and the opportunity function $\widehat{\beta}(q,r)$ is the solution (for β) of:

$$\mu_2(\beta,q) = r \tag{19}$$

4. Viewing $\mu_1(\alpha,q)$ and $\mu_2(\alpha,q)$ as functions of α, properties 1–3 show that μ_1 and μ_2 are "weakly symmetric" about the value $R(q,\widetilde{v})$: the plot of μ_2 vs. α increases monotonically above $R(q,\widetilde{v})$ while μ_1 vs. α decreases monotonically below it. In other words, there exists an increasing function $\zeta(x)$ such that $\zeta(0)=0$ and:

$$\mu_2(\alpha,q) = R(q,\widetilde{v}) + \zeta\left[R(q,\widetilde{v}) - \mu_1(\alpha,q)\right] \tag{20}$$

It sometimes happens that $\zeta(x)=x$, implying that μ_1 and μ_2 are "strongly symmetric" about $R(q,\widetilde{v})$. This will be illustrated in subsequent examples.

Let $\overline{\zeta}(x)$ denote the inverse of the function $\zeta(x)$. That is, $\overline{\zeta}(\zeta(x)) = x$. We can now state the following useful lemma:

Lemma 1 *The opportunity and robustness functions are related as:*

$$\widehat{\beta}(q,r) = \widehat{\alpha}\left(q,\ R(q,\widetilde{v}) - \overline{\zeta}[r - R(q,\widetilde{v})]\right) \tag{21}$$

The immediate use of this lemma is for investigating the partial derivative in eq.(16) of the definition of antagonistic immunities. For notational convenience define the point in (q,r)-space $P = (\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}}),\ R[\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}}),\widetilde{v}] - \overline{\zeta}(r^*))$ where $r^* = r_{\mathrm{w}} - R[\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}}),\widetilde{v}]$. After some manipulation one finds:

$$\frac{\partial\widehat{\beta}(\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}}),r_{\mathrm{w}})}{\partial r_{\mathrm{c}}} = \left(\left.\frac{\partial\widehat{\beta}(q,r_{\mathrm{w}})}{\partial q}\right|_{\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})}\right)^T \frac{\partial\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})}{\partial r_{\mathrm{c}}} \tag{22}$$

$$= \left\{ \left.\underbrace{\frac{\partial\widehat{\alpha}(q,r)}{\partial r}}_{<0}\right|_P \left.\frac{\partial R(q,\widetilde{v})}{\partial q}\right|_{\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})} \underbrace{\left(1 + \left.\frac{\mathrm{d}\overline{\zeta}(x)}{\mathrm{d}x}\right|_{r^*}\right)}_{>0} + \left.\frac{\partial\widehat{\alpha}(q,r)}{\partial q}\right|_P \right\}^T \frac{\partial\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})}{\partial r_{\mathrm{c}}} \tag{23}$$

The quantity denoted '<0' is negative by virtue of theorem 1, and the term marked '>0' is positive because ζ and hence $\overline{\zeta}$ increase monotonically. Eq.(23) provides a means of testing for antagonistic immunities.

Example 3 *Scalar action, q.* If $\partial\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})/\partial r_{\mathrm{c}} > 0$ then the decision maker's maximally robust action increases with his demanded critical reward. This need not be the case, but when it does hold it implies a particular "coherence" between action and expectation. This coherence occurs for instance in eq.(12) of example 1.

When the action is a scalar and when the optimal action $\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})$ is "coherent" with r_{c}, then eq.(23) implies that the robustness and opportunity functions are antagonistic if and only if:

$$\left.\frac{\partial R(q,\widetilde{v})}{\partial q}\right|_{\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})} < \frac{-1}{\left(1 + \left.\frac{\mathrm{d}\overline{\zeta}(x)}{\mathrm{d}x}\right|_{r^*}\right)} \left.\frac{\partial\widehat{\alpha}(q,r)/\partial q}{\partial\widehat{\alpha}(q,r)/\partial r}\right|_P \tag{24}$$

where r^* and P are defined as before. This inequality defines the region of antagonistic immunities in the $(r_{\mathrm{c}},\ r_{\mathrm{w}})$ plane. It not infrequently happens that $\zeta(x)=x$, in which case $\mathrm{d}\overline{\zeta}/\mathrm{d}x = 1$, further simplifying (24). ∎

Example 4 *Strong symmetry of μ_1 and μ_2.* The function $\zeta(x)$ introduced in eq.(20) is not infrequently just $\zeta(x) = x$, which simplifies eqs.(21), (23) and (24). This occurs in example 2. It also occurs if the reward function is $R(q, v) = v^T q$ while the uncertainty is represented by an ellipsoidal info-gap model:

$$\mathcal{C}(\alpha, \widetilde{v}) = \left\{ v : \ (v - \widetilde{v})^T W (v - \widetilde{v}) \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{25}$$

where W is a real, symmetric, positive definite matrix. In this case one finds from eq.(17) that the functions $\mu_n(\alpha, q)$ are:

$$\mu_n(\alpha, q) = \widetilde{v}^T q \pm \alpha \sqrt{q^T W^{-1} q} \tag{26}$$

from which it results that $\zeta(x) = x$. ■

6 Robust Risk-Sensitivity

The lexical definition of risk is a "possibility of loss or injury", a "peril", a "dangerous element or factor" [29]. The etymological origin of the word 'risk' is doubtful, though it may come from earlier words meaning cliff, rock or submarine hill [30]. It is evident that the intuitive pre-scientific concept of risk can be quantified in many ways.

In this section we present a formal definition of robust risk-sensitivity as well as several general results and two more examples. The concept of risk-aversion developed here is non-probabilistic and distinct from the classical Arrow-Pratt theory [18, 25]. Nonetheless, we will show that robust risk sensitivity has interpretations and implications which are consistent with reasonable anticipations from such a theory.

Probabilistic quantification of risk-aversion is feasible in those situations which are informationally rich enough to support probabilistic information. In this paper we consider severe uncertainty as expressed by a non-probabilistic information gap. In this context risk is the potential for lost reward, and the aversion to risk is quantified as an operational sensitivity to environmental uncertainty. The actions of a risk-averse decision maker are strongly constrained by the ambient uncertainty. Conversely, risk-fondness is expressed by the decision maker being able to demand large reward even in the face of a large information gap, implying a wider range of available options. In our context, the aversion to or proclivity for risk is not absolute, but rather a property which emerges in the comparison between different decision makers or alternative strategies.

We will deal with multiple decision makers, whose robustness functions and maximal-robustness actions are denoted $\widehat{\alpha}_i(q, r)$ and $\widehat{q}_i(r)$. The optimal robustness function of decision maker-i is $\widehat{\alpha}_i(\widehat{q}_i(r), r)$. The robustness functions of different decision makers differ because either they have different reward functions, or they use different info-gap uncertainty models, or both.

Definition 2 **Robust risk fondness** *is measured by the robustness function $\widehat{\alpha}(q, r)$.*

Given two decision makers with robustness functions $\widehat{\alpha}_1(q, r)$ and $\widehat{\alpha}_2(q, r)$, we will say that decision maker-2 has greater *robust risk fondness* than decision maker-1 if:

$$\widehat{\alpha}_1(q, r) \le \widehat{\alpha}_2(q, r) \tag{27}$$

Likewise, relation (27) means that decision maker-1 has greater *robust risk aversion* than decision maker-2. The comparison may be applied to all values of r and q, or over a specified domain.

Definition 2 leads immediately to the following result for the optimal robustness functions of two decision makers.

Lemma 2 *If two decision makers have the same set of available actions, $\mathcal{Q}$, then greater robust risk fondness implies greater robust risk fondness at the maximal-robustness action of each decision maker. That is, if eq.(27) holds throughout $\mathcal{Q}$, then:*

$$\widehat{\alpha}_1(\widehat{q}_1(r), r) \le \widehat{\alpha}_2(\widehat{q}_2(r), r) \tag{28}$$

The two decision makers may very well have totally different optimal actions: $\widehat{q}_1(r) \neq \widehat{q}_2(r)$. Nonetheless, if decision maker-2 has greater robust risk fondness than decision maker-1, then lemma 2 asserts that this is true also when each adopts his own optimal action.

The basic decision function in robust rationality, and the most important characterization of the decision maker and his relation to the uncertain environment, is the robustness function, $\widehat{\alpha}(q, r_{\rm c})$, defined in eq.(3). The first gambler's theorem, theorem 1, states that, for fixed action q, the robustness decreases monotonically with the critical reward. Corollary 1/1 states that the robustness function evaluated at the optimal action, $\widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm c})$, is also a monotonically decreasing function in $r_{\rm c}$.

The decreasing robustness function manifests the trade-off between robustness and reward: the decision maker's immunity to uncertainty decreases as greater reward is demanded, as illustrated schematically in fig. 3. Robustness curves such as those in fig. 3 characterize the decision maker as a gambler: courageous or cautious depending on whether he makes great or small demands on the environment, in face of specified vulnerability to uncertainty. The shape and position of the robustness-vs.-reward curve expresses one aspect of the decision maker's sensitivity to risk, as we will now explain.

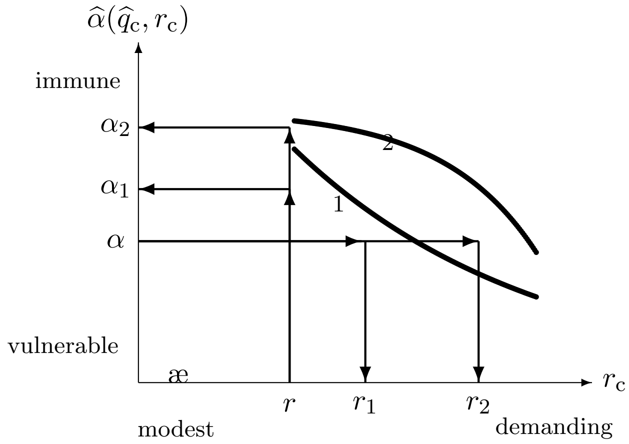

Figure 3: Maximal robustness versus demanded reward for two different decision-makers. Illustrating robust risk sensitivity.

The $r_{\rm c}$ axis in fig. 3 is labelled 'modest' near the origin and 'demanding' away from the origin, and the $\widehat{\alpha}$ axis is labelled 'vulnerable' and 'immune' for small and large values of robustness, respectively. Consider two decision makers, whose robustness functions are shown in the figure: decision maker-2 is relatively immune to uncertainty even when he makes great demands on the environment, while decision maker-1 is relatively vulnerable to uncertainty even when he makes modest demands for reward. The decision makers have different robustness functions because they have different reward functions or different uncertainty models or both.

Decision-maker-2 will look speculative, audacious or daring in the eyes of decision maker-1, while decision maker-1 will appear conservative, cautious, or even timid in decision maker-2's eyes. Consider for instance the arrows rising from the point labeled r on the $r_{\rm c}$ axis. For this value of demanded reward the upper decision maker is relatively immune to uncertainty, while the lower decision maker is relatively vulnerable. In other words, $\alpha_2 > \alpha_1$ so decision maker-2 can tolerate more uncertainty and is less averse to risk than decision maker-1.

Similarly, consider the arrows stemming from the robustness α on the $\widehat{\alpha}$ axis of fig. 3. At this level of uncertainty decision maker-1 is able to demand only a low guaranteed return, r_1, while decision maker-2 can demand greater return r_2. As we will see in connection with theorem 4, this implies that, under particular conditions, decision maker-1 is willing to make a smaller commitment of his resources than decision maker-2. Again, decision maker-1 is showing himself more timid in the face of uncertainty than decision maker-2.

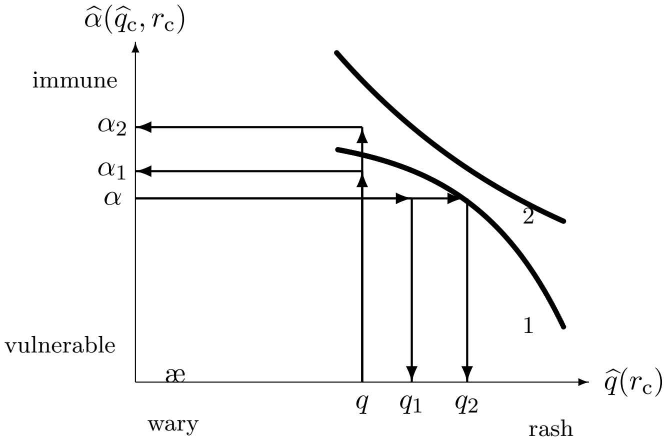

Figure 4: Maximal robustness versus optimal action for two different decision-makers.

It is instructive to consider the variation of robustness $\widehat{\alpha}$ versus the optimal action $\widehat{q}$ (which for now we consider to be a scalar rather than a vector, for simplicity of exposition). Referring to eqs.(2)–(4) we see that, for each choice of the critical reward $r_{\rm c}$, there is an optimal action $\widehat{q}_{\rm c}(r_{\rm c})$, which we will assume to be unique. So, the $r_{\rm c}$-axis of fig. 3 can be transformed to a $\widehat{q}$-axis, as in fig. 4. Sometimes the action can be interpreted as a commitment of money, time, space or some other resource. The action-axis varies from 'wary' at small values of $\widehat{q}$ to 'rash' for large values. Following the arrows from fixed robustness α, one finds $q_2 > q_1$ which means that decision maker-2 is able to make a greater commitment than decision maker-1. Conversely, following the arrows from fixed action q, decision maker-2 enjoys greater immunity to uncertainty than decision maker-1 since $\alpha_2 > \alpha_1$. Again we see that the robustness $\widehat{\alpha}$ serves as a measure of the decision maker's sensitivity to uncertainty and his willingness to take actions which other decision makers would view as either too risky or too timorous.

To summarize the discussion of figs. 3 and 4, we can say that greater robustness to uncertainty implies a willingness for greater commitment of resources at the same level of ambient uncertainty. Equivalently, greater robustness enables the decision maker to demand greater returns from the environment. In other words, increments of robustness-to-uncertainty correspond to increments in fondness for risk. In either fig. 3 or 4, the increment $\alpha_2 - \alpha_1$ can be viewed as a "robustness premium" which decision maker-2 enjoys over decision maker-1. Likewise, in fig. 3 decision maker-2 is able to demand a "reward premium", $r_2 - r_1$, at the same level of immunity to uncertainty. In fig. 4, decision maker-2 is willing to make a greater commitment of resources than decision maker-1. We can use the robustness premium or the reward premium or the commitment premium to compare the risk sensitivities of two decision makers.

The following theorem and its corollaries quantify and illuminate the meaning of these measures of risk sensitivity.

Theorem 3 *Given two decision makers with the same set $\mathcal{Q}$ of available actions, with uniformly continuous reward functions $R_1(q, v)$ and $R_2(q, v)$, with info-gap models of uncertainty $\mathcal{C}_1(\alpha, \widetilde{v})$ and $\mathcal{C}_2(\alpha, \widetilde{v})$, and with finite robustness functions $\widehat{\alpha}_1(q, r)$ and $\widehat{\alpha}_2(q, r)$.* **If:**

$$(1 + \delta)R_1(q, v) \leq R_2(q, v) \tag{29}$$

for some $\delta \geq 0$ and for all $q \in \mathcal{Q}$ and all v in the range of the info-gap models, **and if:**

$$\mathcal{C}_2[(1 + \varepsilon)\alpha, \widetilde{v}] \subseteq \mathcal{C}_1(\alpha, \widetilde{v}) \tag{30}$$

for some $\varepsilon \geq 0$ *and for all* $\alpha \geq 0$,
then:

$$(1+\varepsilon)\widehat{\alpha}_1(q,r) \leq \widehat{\alpha}_2[q,(1+\delta)r] \leq \widehat{\alpha}_2(q,r) \tag{31}$$

with strict inequality on the right if $\delta > 0$, *unless both robustness functions are identically zero.*

Eq.(31) implies that decision maker-2 has greater risk-fondness than decision maker-1:

$$\widehat{\alpha}_1(q,r) \leq \widehat{\alpha}_2(q,r) \tag{32}$$

In particular, if either $\delta > 0$ or $\varepsilon > 0$, then decision maker-2 has **strictly** greater robust risk-fondness than decision maker-1:

$$\widehat{\alpha}_1(q,r) < \widehat{\alpha}_2(q,r) \tag{33}$$

unless both robustness functions are identically zero.

This theorem contains two special cases: (1) identical reward function and nested info-gap models, and (2) ranked reward functions and identical info-gap models. These special cases have particularly important interpretations, so we state them as corollaries 1/3 and corollaries 2/3, without proof.

The extent of our information about uncertain events is represented by an info-gap model. The relation "more information" in one formulation than another is expressed by set-inclusion of the corresponding info-gap models as in relation (30). Model $\mathcal{C}_2$ is more informative than model $\mathcal{C}_1$ because the former constrains the uncertain quantity more tightly than the latter. In $\mathcal{C}_2(\alpha,\widetilde{c})$ the information gap is smaller than in $\mathcal{C}_1(\alpha,\widetilde{c})$. These considerations lead directly to the following corollary to theorem 3.

Corollary 1/3 *Robust risk-aversion increases with uncertainty.* **If** *two decision makers have the same set of available actions, the same uniformly continuous reward function, and nested info-gap models as in eq.(30),* **then** *decision maker-1 has greater robust risk-aversion than decision maker-2.*

The second special case of theorem 3 can be interpreted to mean that, other things being equal, a decision maker's aversion to risk is moderated by the utility he stands to enjoy. The second special case can be expressed as:

Corollary 2/3 *Robust risk-aversion decreases with reward.* **If** *two decision makers have the same set of available actions, the same info-gap model of uncertainty and ranked uniformly continuous reward functions as in eq.(29),* **then** *decision maker-2 has lower robust risk-aversion than decision maker-1.*

Another interesting corollary to theorem 3 is:

Corollary 3/3 *Uncertainty is offset by reward. Given the conditions of theorem 3, there is a new reward function* $R_1^*(q,v)$ *where* $R_1^*(q,v) \geq R_1(q,v)$ *and such that, if* $R_1(q,v)$ *is replaced by* $R_1^*(q,v)$, *then:*

$$\widehat{\alpha}_2(q,r) < \widehat{\alpha}_1^*(q,r) \tag{34}$$

where $\widehat{\alpha}_1^*(q,r)$ *is the robustness function of decision maker-1 based on reward function* $R_1^*(q,v)$.

In other words, the greater uncertainty of decision maker-1 (relation (30)) can be offset by increasing his reward, thereby causing his risk-fondness to exceed that of decision maker-2 (that is, reversing relation (32)).

Example 5 *Risk aversion and plant size.* Fuller and Gerchak [15] study probabilistic risk aversion and the choice of plant size. Their nominal profit function is:

$$R(q) = qR\mathrm{e}^{-\delta L(q)} - c(q) \tag{35}$$

where q is the plant size, R is the revenue per unit plant size, δ is the discount rate, $L(q)$ is the construction lead time and $c(q)$ is the cost of a plant of size q.

Now, instead of a probabilistic uncertainty model, we use an info-gap model. Consider severe lack of information about the discounted revenue per unit plant, whose nominal value is $\widetilde{v}(q) = R\mathrm{e}^{-\delta L(q)}$. Instead of eq.(35), the uncertain profit function is:

$$R(q, v) = qv(q) - c(q) \tag{36}$$

where $v(q)$ is unknown. We have limited prior information which constrains the rate of variation of $v(q)$. One form which this prior knowledge can take is spectral information expressed as bounds on Fourier coefficients. The uncertain function is represented:

$$v(q) = \widetilde{v}(q) + \sum_{n=n_1}^{n_2} \beta_n \left[\sin n\pi q + \cos n\pi q\right] \tag{37}$$

where $\widetilde{v}(q)$ is known and the Fourier coefficients β_n are uncertain. This type of info-gap model of uncertainty is quite common in technological applications. A single Fourier coefficient is used for each mode when the amplitude of each mode is important, but the phase is less so. When both amplitude and phase are important one can introduce two uncertain coefficients for each mode.

Let $N = n_2 - n_1 + 1$, and let $\sigma(q)$ and $\gamma(q)$ be N-vectors of the sines and cosines, respectively, appearing in the sum in eq.(37). Let β be the N-vector of Fourier coefficients. Now we can write eq.(37) more succintly:

$$v(q) = \widetilde{v}(q) + \beta^T \left[\sigma(q) + \gamma(q)\right] \tag{38}$$

The Fourier ellipsoid-bound info-gap model for uncertainty in the discounted revenue is:

$$\mathcal{C}^{(N)}(\alpha, \widetilde{v}) = \left\{ v(q) = \widetilde{v}(q) + \beta^T \left[\sigma(q) + \gamma(q)\right] : \ \beta^T W \beta \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{39}$$

where N is the number of modes in the uncertainty model and W is a real, symmetric, positive definite matrix. After some manipulation one finds the robustness function to be:

$$\widehat{\alpha}(q, r_{\mathrm{c}}) = \frac{1}{\sqrt{\left[\sigma(q) + \gamma(q)\right]^T W^{-1} \left[\sigma(q) + \gamma(q)\right]}} \left(R\mathrm{e}^{-\delta L(q)} - \frac{r_{\mathrm{c}} + c(q)}{q} \right) \tag{40}$$

Now we will illustrate the use of corollary 1/3. For simplicity, let the matrix W in eq.(39) be the identity matrix and let $n_1 = 1$ and $n_2 = N$ in eq.(37). The robustness function, eq. (40), becomes:

$$\widehat{\alpha}(q, r_{\mathrm{c}}) = \frac{1}{\sqrt{\sum_1^N (\sin n\pi q + \cos n\pi q)^2}} \left(R\mathrm{e}^{-\delta L(q)} - \frac{r_{\mathrm{c}} + c(q)}{q} \right) \tag{41}$$

It is readily seen that $\mathcal{C}^{(N)}$ is a more informative uncertainty model than $\mathcal{C}^{(N+1)}$, since:

$$\mathcal{C}^{(N)}(\alpha, \widetilde{v}) \ \subset \ \mathcal{C}^{(N+1)}(\alpha, \widetilde{v}) \tag{42}$$

That is, the uncertainty model $\mathcal{C}^{(N)}(\alpha, \widetilde{v})$ constrains the uncertainty more tightly than $\mathcal{C}^{(N+1)}(\alpha, \widetilde{v})$.

Now, from eq.(41) we see that the robustness function decreases with increasing dimension N. In other words, as stated by corollary 1/3, robust risk aversion increases with uncertainty. Furthermore, eq.(41) shows the marginal increase in risk aversion resulting from a marginal enlargement of the info-gap model through increasing the value of N. ∎

The following theorem establishes a relationship between risk aversion and the size of a decision maker's commitment.

Theorem 4 *Consider two decision makers with the same set $\mathcal{Q}$ of available actions, and whose robustness functions $\widehat{\alpha}_1(q,r)$ and $\widehat{\alpha}_2(q,r)$ both reach unconstrained maxima on $\mathcal{Q}$. The actions are represented by an N-vector q whose elements are denoted q_n, $n = 1, \ldots, N$. Suppose that, for some value r of the critical reward, both robustness functions are positive, smooth and continuous for all $q \in \mathcal{Q}$, that $\widehat{\alpha}_2(q,r)$ is uni-modal in q_n at this r for some particular n, and that $\widehat{\alpha}_1(q,r)$ and $\widehat{\alpha}_2(q,r)$ at this r are related as:*

$$\widehat{\alpha}_1(q,r) = \phi(q)\widehat{\alpha}_2(q,r) \tag{43}$$

for all $q \in \mathcal{Q}$, where $\phi(q)$ is a positive real function. Then, for this value of r, the nth component $\widehat{q}_{1,n}(r)$ of the maximum-robustness action vector for decision maker-1 is less than the nth component $\widehat{q}_{2,n}(r)$ of the maximum-robustness action vector for decision maker-2 if and only if $\phi(q)$ is decreasing in q_n at $\widehat{q}_1(r)$. That is:

$$\widehat{q}_{1,n}(r) \quad \begin{matrix}<\\=\\>\end{matrix} \quad \widehat{q}_{2,n}(r) \quad \text{if and only if} \quad \left.\frac{\partial \phi(q)}{\partial q_n}\right|_{q=\widehat{q}_1(r)} \quad \begin{matrix}<\\=\\>\end{matrix} \quad 0 \tag{44}$$

The basic import of theorem 4 is that greater robust risk-fondness, by itself, does not necessarily imply greater optimal action. If $\phi(q)$ in eq.(43) is less than unity, then decision maker-1 is more risk-averse than decision maker-2. The theorem establishes that this does not guarantee that decision maker-1's actions will take smaller magnitude than those of decision maker-2. This by itself is not surprising since the action vector q may contain decisions, such as the choice of a geometrical dimension or the time of an event, for which 'big' does not imply any subjectively greater commitment. Hence, the interesting part of theorem 4 is that it establishes a simple criterion by which to determine when greater risk-aversion *does* imply smaller values of the optimal actions. We elaborate this in the following corollaries.

Corollary 1/4 *With the conditions of theorem 4 and also that $\widehat{\alpha}_1(q,r)$ is uni-modal in q_n at this r, the maximum-robustness actions of the two decision makers are ranked according to the logarithmic derivatives of their optimal robustness functions:*

$$\widehat{q}_{1,n}(r) \quad \begin{matrix}<\\=\\>\end{matrix} \quad \widehat{q}_{2,n}(r) \quad \text{if and only if} \quad 0 \quad \begin{matrix}<\\=\\>\end{matrix} \quad \left.\frac{\partial \ln \widehat{\alpha}_2(q,r)}{\partial q_n}\right|_{q=\widehat{q}_1(r)} \tag{45}$$

Equivalently:

$$\widehat{q}_{1,n}(r) \quad \begin{matrix}<\\=\\>\end{matrix} \quad \widehat{q}_{2,n}(r) \quad \text{if and only if} \quad 0 \quad \begin{matrix}>\\=\\<\end{matrix} \quad \left.\frac{\partial \ln \widehat{\alpha}_1(q,r)}{\partial q_n}\right|_{q=\widehat{q}_2(r)} \tag{46}$$

If a decision maker's robustness function achieves unconstrained maxima on the set of available actions, then the logarithmic derivatives of the robustness, $\partial \ln \widehat{\alpha}(q,r)/\partial q_n$, equal zero at the optimal action. Consequently, we can roughly paraphrase corollary 1/4 as follows: *greater logarithmic derivative of the robustness implies greater optimal action.*

Example 6 *Risk aversion and plant size: continued.* To illustrate the use of corollary 1/4 let us continue example 5 with the following numerical formulation. Let $c(q) = a\sqrt{q}$ and $L(q) = b_1 q + b_0$, so the lead-time increases with b_1. The parameters of the nominal discount rate are $\delta = 0.08$ and $R\mathrm{e}^{-\delta b_0} = 3a$. Fig. 5 illustrates that the robustness functions are indeed uni-modal, as required by theorem 4. (Note however that the lower function is not concave. Concavity, which is a stronger condition than uni-modality, is not required by theorem 4.) Fig. 6 shows the logarithmic derivative

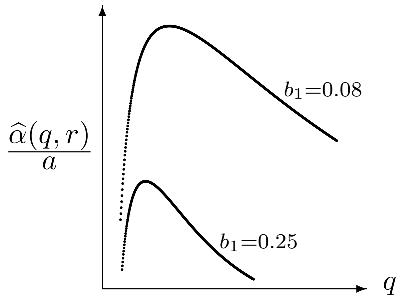

Figure 5: Robustness versus the action, for example 6. $r_c/a = 25$.

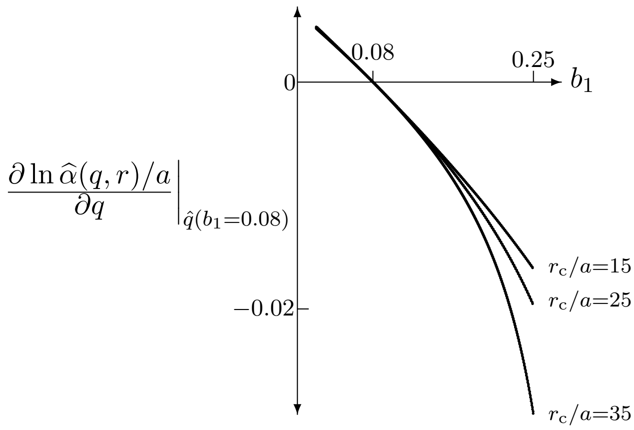

Figure 6: Logarithmic derivative of the robustness at the optimal action for $b_1 = 0.08$, versus b_1, for various values of r_c/a. For example 6

of the robustness function versus the lead-time parameter b_1. (Recall that the construction lead-time increases linearly with b_1.) The three curves are for different values of the critical reward r_c/a. For each curve, the derivative is evaluated at the optimal action $\widehat{q}$ for lead-time parameter $b_1 = 0.08$. So of course the derivatives vanish at $b_1 = 0.08$. (The optimal actions at $b_1 = 0.08$ for these three values of r_c are $\widehat{q} = 34.11$, 44.27 and 53.21 for $r_c/a = 15$, 25 and 35, respectively.)

We see from fig. 6 that the derivatives are negative for longer construction lead-time ($b_1 > 0.08$), and positive for shorter lead-time. The interpretation of this sign-trend is provided by corollary 1/4: the optimal action (optimal plant size) decreases with lead time. For example, eq.(45) states that $\widehat{q}_1 < \widehat{q}_2$ if and only if $0 < \partial \ln \widehat{\alpha}_2 / \partial q$ at $q = \widehat{q}_1$. If decision maker-1 is considering a lead-time coefficient of $b_1 = 0.08$ while decision maker-2 is considering a different lead-time, then negative derivatives of decision maker-2's robustness, evaluated at decision maker-1's optimal action, imply lower optimal plant size for decision maker-2. Likewise, positive derivatives imply greater optimal plant size for decision maker-2.

It is also interesting to note that, at large lead-times, the magnitude of the derivative increases with the critical reward. That is, demanding greater return ultimately forces much smaller investment when the lead-time is large, because of the decision maker's aversion to risk.■

We have already explained that theorem 4 shows that enhanced robust risk-aversion by itself does not imply reduced optimal action. However, the following corollary establishes a condition which, in addition to robust risk-aversion, *does* imply ranking of the optimal actions.

Corollary 2/4 *With the conditions of theorem 4, greater robust risk-fondness implies greater optimal action if and only if the robustness-increment is increasing in q at the risk-averse optimum. That is, given that $\phi(q) < 1$, then:*

$$\widehat{q}_{1,n}(r) \quad \begin{matrix} < \\ = \\ > \end{matrix} \quad \widehat{q}_{2,n}(r) \quad \text{if and only if} \quad \left. \frac{\partial\left[\widehat{\alpha}_2(q,r) - \widehat{\alpha}_1(q,r)\right]}{\partial q_n} \right|_{q=\widehat{q}_1(r)} \quad \begin{matrix} > \\ = \\ < \end{matrix} \quad 0 \tag{47}$$

7 Summary and Discussion

In this paper we have presented a methodology for decisions with severe uncertainty which is conceptually quite different from classical probabilistic thought, and we have explored risk-taking behavior when using this methodology. The use of info-gap uncertainty-models for quantifying severe uncertainty may be motivated by the nature of that uncertainty itself: a stark lack of data, an "information gap" between what *is known* and what *needs to be known* in order to make an optimal decision. Severe uncertainty arises in strategic planning, in advanced engineering design, in micro-economics and other areas. When information is very sparse one is hard put to verify a probabilistic model, while an info-gap model can be formulated as a family of nested sets consistent with available data. Once the info-gap model is adopted, the desire for uncertainty-avoidance leads naturally to the search for strategies which maximize the decision-maker's immunity to uncertainty, while at the same time guaranteeing a specified minimum reward. This strategy, implemented with info-gap models, is what we have called 'robust rationality'.

In section 4 we proved a "gambler's theorem" which quantifies the trade-off between reward and immunity-to-uncertainty. This trade-off forces the decision-maker to gamble, but without requiring any probabilistic thinking on his part. This theorem epitomizes the method of robust rationality, which is based on non-probabilistic set-models of information-gap uncertainty. Also, this theorem highlights the robustness function $\widehat{\alpha}(q, r_{\rm c})$ as a decision tool, showing how the analysis of uncertainty may lead the decision maker to modify his preferences for reward and immunity. Theorem 2 is complementary to theorem 1, and uses the opportunity function $\widehat{\beta}(q, r_{\rm w})$ to express the trade-off between certainty and windfall reward.

In section 5 we explored the antagonism between robustness-to-failure and opportunity-for-success. Eq.(23) establishes the basis for investigating whether this antagonism occurs.

In section 6 we employed the robustness function as a measure of risk-sensitivity. $\widehat{\alpha}(q, r)$ is the greatest gap in the decision maker's information which is consistent with attaining reward no less than r, if he implements the strategy q. The optimal strategy, $\widehat{q}_c(r)$, maximizes the decision maker's immunity, and $\widehat{\alpha}(\widehat{q}_c(r), r)$ is the greatest possible robustness. The character of a decision maker as either a wary or a reckless gambler is embedded in his robustness function. The robustness functions of two different decision makers reveal the different degrees to which these individuals will thwart uncertainty to make demands on the environment. The robustness function expresses both the decision maker's prior preferences and the extent of his ignorance of the environment. The robustness function quantifies the trade-offs between certainty and reward which are entailed by the decision maker's reward function and uncertainty model. Theorem 3 and its corollaries illuminate some properties of the robust measure of risk sensitivity. Theorem 4 and its corollaries establish conditions in which the magnitude of the decision maker's commitment of resources will increase with his fondness for risk.

The relation between the concept of risk-sensitivity developed here and the classical ideas of risk-aversion developed by Arrow, Pratt and others in economics can be understood in terms of the underlying models of uncertainty. An uncertainty model — whether probabilistic, fuzzy, info-gap, or whatever — generates a decision theory. A decision theory, in turn, enables characterization of a decision maker. Thus the Arrow-Pratt assessment of risk aversion employs the von Neumann-Morgenstern utility function, which itself is a probabilistic entity. The risk-assessment discussed in this paper employs the robustness function which is meaningful in the context of info-gap modelling of uncertainty. From the axiomatic point of view, these approaches are incompatible [7]. From the practical point of view, nothing prevents an analyst from adopting whatever methodology matches the type and extent of information available. In fact, hybridization is even possible, as explored elsewhere [6].

Uncertainty appears in many different guises, and the methods for its modelling and management must show the same versatility. The implication for decisions under uncertainty is that no single concept, method or theory will match all decision problems fraught with uncertainty. One should expect a plethora of fundamentally conflicting tools, combined or selected in each specific application according to the nature of the information and the demands of the analysis.

The engineering experience of recent decades has shown a blossoming of non-probabilistic theories of uncertainty, which have grown from the exigencies of a range of fundamentally different types and tendencies of uncertain phenomena. Fuzzy logic is the most popular deviation from probabilistic orthodoxy, and it has spawned a litter of fruitful variations in the form of 'possibility theory', 'necessity theory', and other axiomatic variations on Kolmogorov's probability [14]. The neural network analog of biological systems has also gained great popularity due to its dexterity in absorbing and exploiting new information in highly unstructured environments [17]. Convex models of uncertainty, and the more general information-gap set-models of uncertainty, are another quite radical departure from the probabilistic tradition.

In this paper we have developed concepts for decision and analysis based on the idea of immunity-to-uncertainty. Without impeaching either classical probabilistic tools or more modern developments such as fuzzy logic, we have developed a non-probabilistic approach based on information-gap models of uncertainty which may fit the needs of analysts and decision makers in some situations. Set-models of uncertainty, and in particular info-gap models such as convex models, are apposite in decision-environments plagued by severe lack of information.

8 Appendix: Proofs.

The following lemma will be used in the proofs of theorems 1 and 3.

Lemma 3 *If $R(q,v)$ is uniformly continuous and if the robustness function $\widehat{\alpha}(q,r)$ exists (that is, it is finite), then:*

$$\min_{v \in \mathcal{C}(\widehat{\alpha}(q,r),\widetilde{v})} R(q,v) = r \tag{48}$$

That is, if $\widehat{\alpha}(q,r)$ is the robustness for critical reward r, then the minimal reward on an info-gap model of size $\widehat{\alpha}(q,r)$ equals r.

Proof of lemma 3. By eq.(2):

$$\min_{v \in \mathcal{C}(\widehat{\alpha}(q,r),\widetilde{v})} R(q,v) \geq r \tag{49}$$

Suppose however that strict inequality holds:

$$\min_{v \in \mathcal{C}(\widehat{\alpha}(q,r),\widetilde{v})} R(q,v) > r \tag{50}$$

Since $R(q,v)$ is uniformly continuous and $\mathcal{C}(\alpha,\widetilde{v})$ is a family of nested sets, there is a $\beta > \widehat{\alpha}(q,r)$ such that:

$$\min_{v \in \mathcal{C}(\beta,\widetilde{v})} R(q,v) > r \tag{51}$$

But this is a contradiction, since $\widehat{\alpha}(q,r)$ is the maximum of the set $\mathcal{A}(q,r)$. Thus, instead of (49) we have the equality in eq.(48) which proves the lemma. ∎

Proof of theorem 1. (1) Since $\widehat{\alpha}(q,r_1)$ and $\widehat{\alpha}(q,r_2)$ exist and are positive, the α-sets $\mathcal{A}(q,r_1)$ and $\mathcal{A}(q,r_2)$ are bounded and not empty. Let $\alpha \in \mathcal{A}(q,r_2)$, which means that:

$$\min_{v\in\mathcal{C}(\alpha,\widetilde{v})} R(q,v) \geq r_2 > r_1 \tag{52}$$

From this we conclude that the same α also belongs to $\mathcal{A}(q,r_1)$. Hence:

$$\mathcal{A}(q,r_2) \subseteq \mathcal{A}(q,r_1) \tag{53}$$

$\widehat{\alpha}(q,r_n)$ is the least upper bound of $\mathcal{A}(q,r_n)$, so (53) implies that:

$$\widehat{\alpha}(q,r_1) \geq \widehat{\alpha}(q,r_2) \tag{54}$$

(2) By lemma 3 we know that:

$$\min_{v\in\mathcal{C}(\widehat{\alpha}(q,r_1),\widetilde{v})} R(q,v) = r_1 \tag{55}$$

By a similar argument we conclude that:

$$\min_{v\in\mathcal{C}(\widehat{\alpha}(q,r_2),\widetilde{v})} R(q,v) = r_2 \tag{56}$$

(3) Now suppose that, in addition to relation (54), the following holds:

$$\widehat{\alpha}(q,r_1) = \widehat{\alpha}(q,r_2) \tag{57}$$

But, by the supposition of the theorem, the righthand sides of eqs.(55) and (56) are different ($r_1 < r_2$). Hence the lefthand sides of these equations must also differ. Thus $\widehat{\alpha}(q,r_1)$ and $\widehat{\alpha}(q,r_2)$ can not be equal and the supposition in (57) is false and the strict inequality in relation (8) results from (54). ■

Proof of corollary 1/1. By theorem 1:

$$\widehat{\alpha}(\widehat{q}_{\rm c}(r_2),r_1) > \widehat{\alpha}(\widehat{q}_{\rm c}(r_2),r_2) \tag{58}$$

But, by the definition of maximum robustness, eq.(4):

$$\widehat{\alpha}(\widehat{q}_{\rm c}(r_1),r_1) \geq \widehat{\alpha}(\widehat{q}_{\rm c}(r_2),r_1) \tag{59}$$

The last two relations prove the desired result. ■

Proof of theorem 2. Since $\zeta(x)$ is an increasing function, theorem 2 results immediately from lemma 1 (eq.(21)) and theorem 1 (eq.(8)). ■

Proof of corollary 1/2. By theorem 2:

$$\widehat{\beta}(\widehat{q}_{\rm w}(r_2),r_1) < \widehat{\beta}(\widehat{q}_{\rm w}(r_2),r_2) \tag{60}$$

But, by the definition of optimum opportunity, eq.(7):

$$\widehat{\beta}(\widehat{q}_{\rm w}(r_1),r_1) \leq \widehat{\beta}(\widehat{q}_{\rm w}(r_2),r_1) \tag{61}$$

The last two relations prove the desired result. ■

Proof of lemma 1. Eq.(19) states that:

$$\mu_2(\widehat{\beta}(q,r),q) = r \tag{62}$$

From eq.(20) we can replace the lefthand side of eq.(62) as:

$$R(q,\widetilde{v}) + \zeta\left[R(q,\widetilde{v}) - \mu_1(\widehat{\beta}(q,r),q)\right] = r \tag{63}$$

Re-arranging and inverting the function ζ leads to:

$$\mu_1(\widehat{\beta}(q,r),q)) = R(q,\widetilde{v}) - \overline{\zeta}(r - R(q,\widetilde{v})) \tag{64}$$

Now, eq.(18) states that:

$$\mu_1(\widehat{\alpha}(q,r),q) = r \tag{65}$$

Comparing this with eq.(64) we obtain relation (21). ■

Proof of lemma 2. Since decision maker-2 is fonder of risk than decision maker-1 throughout $\mathcal{Q}$, eq.(27) holds at the optimal action for decision maker-1:

$$\widehat{\alpha}_1(\widehat{q}_1(r),r) \le \widehat{\alpha}_2(\widehat{q}_1(r),r) \tag{66}$$

Eq.(4) states that the optimal action $\widehat{q}_2(r)$ maximizes the robustness function for decision maker-2, so:

$$\widehat{\alpha}_2(\widehat{q}_1(r),r) \le \widehat{\alpha}_2(\widehat{q}_2(r),r) \tag{67}$$

Combining the last two relations concludes the proof. ■

Proof of Theorem 3. From lemma 3 we know that:

$$\min_{v\in\mathcal{C}_1(\widehat{\alpha}_1(q,r),\widetilde{v})} R_1(q,v) = r \tag{68}$$

For convenience let $\widehat{\alpha}_1 = \widehat{\alpha}_1(q,r)$. In light of relations (29) and (30), we conclude that:

$$\min_{v\in\mathcal{C}_2[(1+\varepsilon)\widehat{\alpha}_1,\widetilde{v}]} \frac{1}{1+\delta} R_2(q,v) \ge \min_{v\in\mathcal{C}_1(\widehat{\alpha}_1,\widetilde{v})} R_1(q,v) = r \tag{69}$$

Hence:

$$\min_{v\in\mathcal{C}_2[(1+\varepsilon)\widehat{\alpha}_1,\widetilde{v}]} R_2(q,v) \ge (1+\delta)r \tag{70}$$

So, referring to eqs.(2) and (3), we conclude that the robustness of decision maker-2, for critical reward $(1+\delta)r$, is no less than $(1+\varepsilon)\widehat{\alpha}_1$:

$$(1+\varepsilon)\widehat{\alpha}_1(q,r) \le \widehat{\alpha}_2[q,(1+\delta)r] \tag{71}$$

Also, by the gambler's theorem:

$$\widehat{\alpha}_2[q,(1+\delta)r] \le \widehat{\alpha}_2(q,r) \tag{72}$$

with strict inequality if $\delta > 0$. The last two relations complete the proof of relation (31). ■

Proof of Corollary 3/3. Let $\widehat{\alpha}_1$ denote the robustness for decision maker-1 using reward function R_1, and let $\widehat{\alpha}_1^*$ denote his robustness with reward function R_1^*. By lemma 3:

$$\min_{v\in\mathcal{C}_1(\widehat{\alpha}_1(q,r),\widetilde{v})} R_1 = r \tag{73}$$

By theorem 3:

$$\widehat{\alpha}_1(q,r) \le \widehat{\alpha}_2(q,r) \tag{74}$$

so:

$$\min_{v\in\mathcal{C}_1(\widehat{\alpha}_2,\widetilde{v})} R_1 \le \min_{v\in\mathcal{C}_1(\widehat{\alpha}_1,\widetilde{v})} R_1 \tag{75}$$

Let Δ denote the excess of the righthand over the lefthand side of (75), which is finite since $\widehat{\alpha}_1$ exists. Now choose decision maker-1's new reward function R_1^* so that:

$$R_1^* > \Delta + R_1 \tag{76}$$

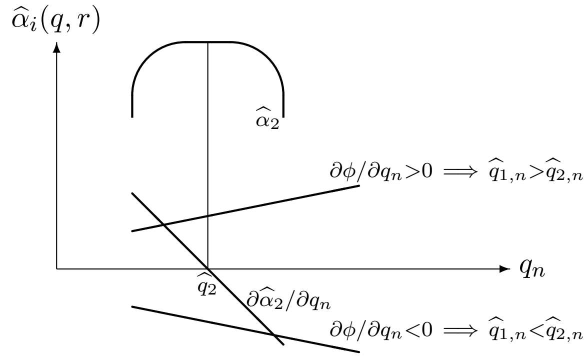

Figure 7: Illustration for proof of theorem 4.

Then:

$$\min_{v\in\mathcal{C}_1(\widehat{\alpha}_2,\widetilde{v})} R_1^* > \Delta + \min_{v\in\mathcal{C}_1(\widehat{\alpha}_2,\widetilde{v})} R_1 = r \tag{77}$$

So, we conclude that decision maker-1's new robustness at critical reward r is greater than $\widehat{\alpha}_2$:

$$\widehat{\alpha}_1^*(q,r) > \widehat{\alpha}_2(q,r) \tag{78}$$

which completes the proof. ■

Proof of theorem 4. (See fig. 7). Since the maxima of the robustness functions are unconstrained, they occur when the first derivatives of $\widehat{\alpha}_i$ vanish. So, using eq.(43), $\widehat{q}_1(r)$ is the solution for q of the following N relations:

$$0 = \frac{\partial\phi}{\partial q_n}\widehat{\alpha}_2 + \phi\frac{\partial\widehat{\alpha}_2}{\partial q_n}, \quad n = 1, \ldots, N \tag{79}$$

Since $\widehat{\alpha}_2$ and ϕ are both positive, the N expressions on the right vanish (producing an extremum for $\widehat{\alpha}_1(q,r)$) only if $\partial\phi/\partial q_n$ and $\partial\widehat{\alpha}_2/\partial q_n$ differ in sign or both vanish. Since $\widehat{\alpha}_2(q,r)$ is uni-modal in q_n for this value of r, $\partial\widehat{\alpha}_2/\partial q_n$ is positive for values of q_n less than $\widehat{q}_{2,n}$, and $\partial\widehat{\alpha}_2/\partial q_n$ is negative for values of q greater than $\widehat{q}_{2,n}$. If $\partial\phi/\partial q_n$ is negative at $\widehat{q}_{1,n}(r)$ then $\partial\widehat{\alpha}_2/\partial q_n$ must be positive at $\widehat{q}_{1,n}(r)$, implying that $\widehat{q}_{1,n}$ is to the left of $\widehat{q}_{2,n}$. Likewise, if $\partial\phi/\partial q_n$ is positive at $\widehat{q}_{1,n}(r)$ then $\partial\widehat{\alpha}_2/\partial q_n$ must be negative at $\widehat{q}_{1,n}(r)$, implying that $\widehat{q}_{1,n}$ is to the right of $\widehat{q}_{2,n}$. This concludes the proof. ■

Proof of Corollary 1/4. Since $\phi = \widehat{\alpha}_1/\widehat{\alpha}_2$ and since both robustness functions are strictly positive, one can readily show that the righthand relation (44) can be expressed differently:

$$\left.\frac{\partial\phi(q)}{\partial q_n}\right|_{q=\widehat{q}_1(r)} \begin{array}{c}<\\=\\>\end{array} 0 \quad \text{if and only if} \quad \left.\frac{\partial\ln\widehat{\alpha}_1(q,r)}{\partial q_n}\right|_{q=\widehat{q}_1(r)} \begin{array}{c}<\\=\\>\end{array} \left.\frac{\partial\ln\widehat{\alpha}_2(q,r)}{\partial q_n}\right|_{q=\widehat{q}_1(r)} \tag{80}$$

We note that both derivatives are evaluated at the optimal action of decision maker-1. In other words, relation (44) in theorem 4 becomes:

$$\widehat{q}_{1,n}(r) \begin{array}{c}<\\=\\>\end{array} \widehat{q}_{2,n}(r) \quad \text{if and only if} \quad \left.\frac{\partial\ln\widehat{\alpha}_1(q,r)}{\partial q_n}\right|_{q=\widehat{q}_1(r)} \begin{array}{c}<\\=\\>\end{array} \left.\frac{\partial\ln\widehat{\alpha}_2(q,r)}{\partial q_n}\right|_{q=\widehat{q}_1(r)} \tag{81}$$

But, from the symmetry of the robustness functions, a similar argument leads to:

$$\widehat{q}_{1,n}(r) \begin{array}{c}<\\=\\>\end{array} \widehat{q}_{2,n}(r) \quad \text{if and only if} \quad \left.\frac{\partial\ln\widehat{\alpha}_2(q,r)}{\partial q_n}\right|_{q=\widehat{q}_2(r)} \begin{array}{c}>\\=\\<\end{array} \left.\frac{\partial\ln\widehat{\alpha}_1(q,r)}{\partial q_n}\right|_{q=\widehat{q}_2(r)} \tag{82}$$

Since the robustness functions achieve unconstrained maxima on $\mathcal{Q}$ we see that:

$$0 = \left. \frac{\partial \ln \widehat{\alpha}_2(q,r)}{\partial q_n} \right|_{q=\widehat{q}_2(r)} = \left. \frac{\partial \ln \widehat{\alpha}_1(q,r)}{\partial q_n} \right|_{q=\widehat{q}_1(r)} \tag{83}$$

which completes the proof. ■

Proof of Corollary 2/4. Denote $\Delta\widehat{\alpha} = \widehat{\alpha}_2 - \widehat{\alpha}_1$. From eq.(43) we have:

$$\frac{\partial \Delta\widehat{\alpha}}{\partial q_n} = \frac{\partial \widehat{\alpha}_2}{\partial q_n}(1-\phi) - \widehat{\alpha}_2 \frac{\partial \phi}{\partial q_n} \tag{84}$$

$\widehat{\alpha}_2$ is positive by supposition, and $1-\phi$ is positive because decision maker-2 has greater robust risk fondness than decision maker-1. When evaluated at $\widehat{q}_1(r)$, the two derivatives on the righthand side of eq.(84) are either both zero or they are of opposite sign, as explained in the proof of theorem 4 in connection with eq.(79). Consequently, the righthand side of eq.(84) is positive if and only if $\partial\phi/\partial q_n$ is negative, and negative if and only if $\partial\phi/\partial q_n$ is positive. Hence we see that $\partial\Delta\widehat{\alpha}/\partial q_n$ is positive, zero, or negative when $\partial\phi/\partial q_n$ is negative, zero, or positive, respectively. The assertion of the corollary now results from theorem 4. ■

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