ASCE *Journal of Structural Engineering,* to appear

Redundancy and Robustness, Or, When is Redundancy Redundant?

Yoshihiro Kanno[1] and Yakov Ben-Haim[2]

Abstract

The redundancy of a structure refers to the extent of degradation which the structure can suffer without losing some specified elements of its functionality. However, since future structural degradation is unknown during design and analysis, it is evident that structural redundancy is related to robustness against uncertainty. We propose a quantitative and widely applicable concept of "strong redundancy" and show its relation to the info-gap robustness of the structure. In particular, one of our propositions establishes general conditions in which the strong redundancy is equivalent to the robustness. We also define a concept of "weak redundancy" and present propositions which relate it to the strong redundancy and the robustness. We illustrate our results with several heuristic and engineering examples.
Keywords: Robustness, redundancy, info-gap, uncertainty, structural integrity, structural degradation.

1 Introduction

The concept of redundancy in structures is central in many design philosophies, and its importance has long been recognized by structural engineers. Several definitions of redundancy have been proposed, for example in terms of the collapse load, the number of plastic hinges, the probability of system failure, etc. However, the definition of redundancy still remains controversial. This paper presents a definition and a quantitative measure of redundancy. Then, using info-gap theory, we provide a measure of robustness and relate it to the concept of redundancy.

Roughly speaking, the redundancy of a structure refers to the extent of degradation which the structure can suffer without losing some specified elements of its functionality. For example, design in a seismically active region may require that life-protecting functionality be preserved after an earthquake, but not necessarily inhabitability of the structure. Or, in contrast, a structure is not considered as being redundant if the failure of one structural component immediately causes the failure of the entire structure and loss of all functional attributes. This paper proposes a precise rendition of this intuitive idea and connects it to robustness against uncertainty.

Survey. Probably the most classical measure of the redundancy of a truss or frame, the *degree of static indeterminacy*, is defined by:

$$s = N - \mathrm{rank} H, \tag{1}$$

where $H \in \Re^{d \times N}$ is the equilibrium matrix (relating external forces to axial forces, shears and bending moments in the elements), N is the number of internal forces, and d is the number of degrees of freedom of displacements. Although s is called the degree of redundancy by some authors, it has been pointed out by Frangopol and Curley (1987) and Pandey and Barai (1997) that s is not always suitable for assessing the performance of the entire structural system.

Frangopol and Curley (1987) defined the *strength redundant factor* r as

$$r = \frac{l_{\mathrm{intact}}}{l_{\mathrm{intact}} - l_{\mathrm{damaged}}}, \tag{2}$$

[0]\people\kanno\Rbs-Redun\rr09.tex 10.8.2010

[1]Department of Mathematical Informatics, Graduate School of Information Science and Technology, The University of Tokyo, Tokyo 113-8656, Japan, kanno@mist.i.u-tokyo.ac.jp

[2]Yitzhak Moda'i Chair in Technology and Economics, Faculty of Mechanical Engineering, Technion — Israel Institute of Technology, Haifa 32000 Israel, yakov@technion.ac.il

where l_{intact} is the ultimate strength of the intact (undamaged) structure, and l_{damaged} is that of the damaged structure. For example, Ohi *et al.* (2004) considered the plastic limit load factor to assess the ultimate strength of the structure, and used the inverse of r to represent the effect of the deficiency of a structural component; in this setting, l_{damaged} is the limit load factor after a member is removed from a structure. The redundancy measure in eq.(2) has also been extended for probabilistic uncertainty by Fu and Frangopol (1990) and Okasha and Frangopol (2009):

$$r = \frac{P(D) - P(C)}{P(C)},$$

where $P(C)$ is the probability of the system collapse, and $P(D)$ is the probability of the failure of a structural component. Hendawi and Frangopol (1994) define $P(C)$ and $P(D)$ as the probability of collapse of the intact structure and the probability that any first-member-yielding occurs in the intact structure, respectively.

The residual strength index is defined by Feng and Moses (1986) as:

$$\frac{l_i}{l_{\text{u}}}, \tag{3}$$

where l_{u} is the ultimate strength of the structural system, and l_i is the strength of the structural system after the ith structural component has failed. Specifically, for a frame structure, the decrease of the linear buckling factor due to deficiency of members was investigated numerically by Shafer and Bajpai (2005). Husain and Tsopelas (2004) define the redundancy-strength index as the inverse of eq.(3), i.e., $l_{\text{u}}/l_{\text{y}}$, where l_{y} is the strength of the structure at the point of the first "significant yielding". A redundancy measure based on a sensitivity coefficient proposed by Pandey and Barai (1997) is:

$$\frac{v_i}{(\partial g_j(p)/\partial p_i)},$$

where g_j is the structural response, $p = (p_i)$ is the vector of design variables, and v_i is the volume of the ith structural component.

For the earthquake-resistant design of frame structures, the number of plastic hinges which emerge when the structure collapses is used for evaluating the redundancy by Bertero and Bertero (1999) and Tesfamariam and Saatcioglu (2010).

Paliou, Shinozuka and Chen (1990) used the conditional probability $P(S|E)$ to assess the redundancy, which is defined as the (conditional) probability that the structure will eventually survive (S), given the event E, which is the (simultaneous) failure of some members. For assessing the redundancy against earthquakes, the uniform redundancy factor was proposed by Liao, Wen and Foutch (2007) and Wen and Song (2003) based on the probability of incipient collapse. Redundancy measures based on Shannon's information entropy are discussed by Hoshiya and Yamamoto (2002, 2003), Au (2003), and Žiha (2000). Hendawi and Frangopol (1994) considered the gap between the load at which the member begins to collapse and the load when the entire structure collapses.

Aim. This paper proposes an intuitive and quantitative concept of redundancy which is relevant to a wide range of structures and applications. Our development has two specific goals.

The first aim relates to the management of uncertainty. Redundancy is evaluated with respect to failure of structural components. For example, high redundancy often means that the structure suffers only small degradation of performance when a structural component fails. However, for a real-world structure we do not know in advance how many and which components will fail. In other words, redundancy is related to robustness against uncertainty. We will establish a quantitative connection between redundancy and robustness against uncertainty.

The second goal of this paper relates to the performance requirements of a structure. A structure has large redundancy if a performance requirement is satisfied for any of a range of possible failures. Thus the notion of a constraint—a performance requirement—over a set of contingencies is central to the idea of redundancy. The concept of redundancy developed in this paper is applicable to a very broad range of performance requirements.

Organization. In section 2 we define a concept called strong redundancy and discuss its intuitive meaning. In section 3 we define a concept of robustness to uncertainty. In section 4 we explore the relation between strong redundancy and robustness, proving two propositions. The first establishes that the strong redundancy is bounded below by the robustness, and the second establishes, under somewhat stricter but still fairly general conditions, that in fact strong redundancy and robustness are identical. Sections 5 and 6 discuss two engineering examples of strong redundancy. In section 7 we introduce a concept called weak redundancy and prove two propositions which establish relations between weak redundancy, strong redundancy and robustness. We conclude in section 8. All proofs appear in section 9.

2 Strong Redundancy

2.1 The System and Its Performance Requirement

Consider a system whose performance $g(p)$ depends on a vector p which specifies the system. Let $\mathcal{T}$ denote the class of all p values of relevant or possible realizations of the system. A small value of $g(p)$ is preferred over a large value. The performance requirement is that $g(p)$ not exceed a critical or maximal allowed value, $\overline{g}$:

$$g(p) \le \overline{g} \tag{4}$$

Define $|p| = \sum_{i=1}^{N} |p_i|$ and let $p \le p'$ mean that $p_i \le p'_i$ for $i = 1, \ldots, N$. When $p \le p'$ we will say that the system represented by p is no stronger (usually, actually weaker) than the system represented by p'.

Example 1 Consider a collection of nodes between which can be connected up to N bars with frictionless joints to form a truss with specified boundary conditions. Let p be a binary indicator vector whose elements equal either 0 or 1: $p_i = 1$ means that the ith bar is present while $p_i = 0$ means that the ith bar is absent, for $i = 1, \ldots, N$. Let $\mathcal{T}$ denote this set of possible trusses. The value of $g(p)$ expresses whether or not the system is statically stable. A small value of $g(p)$ implies stability. ∎

Example 2 As in example 1, consider a collection of nodes between which can be connected up to N bars with frictionless joints to form a truss with specified boundary conditions. Let $p \in \Re_{+}^{N}$ be a non-negative real vector whose ith element is the cross-sectional area of the ith bar. Let $\mathcal{T}$ denote this set of possible trusses. The value of $g(p)$ is the compliance of the system. A small value of $g(p)$ is desirable. ∎

2.2 Strong Redundancy: Definition

We now define a concept of "strong redundancy". To do this we first introduce a "deficiency set". Let $\mathcal{D}(\alpha, \widetilde{p})$ be the set of all structures which are deficient, by an amount α, with respect to the nominal structure $\widetilde{p}$. The systems in $\mathcal{D}(\alpha, \widetilde{p})$ have suffered a decrement in structural integrity, with respect to the designed system $\widetilde{p}$, by an amount α. Because we do not know the extent of structural damage which will occur in the future we will be refer to α as the horizon of uncertainty. Let us consider several examples of deficiency sets before presenting a formal definition of strong redundancy.

Example 3 In continuation of example 1 let the deficiency set $\mathcal{D}(\alpha, \widetilde{p})$ be the set of all trusses in $\mathcal{T}$ which differ from the nominal design by removing exactly α bars:

$$\mathcal{D}(\alpha, \widetilde{p}) = \{p : \ p \in \mathcal{T}, \ p \le \widetilde{p}, \ |p - \widetilde{p}| = \min[\alpha, \ |\widetilde{p}|]\}, \quad \alpha = 0, 1, 2, \ldots \tag{5}$$

Note that, when the horizon of uncertainty equals or exceeds the norm of $\widetilde{p}$, (which equals the number of bars in the nominal structure), then exactly $|\widetilde{p}|$ components are lacking from $\widetilde{p}$ so the truss is now completely empty. Consequently $\mathcal{D}(\alpha, \widetilde{p})$ remains constant for all $\alpha \ge |\widetilde{p}|$. In fact, this deficiency set will contain only the zero element. ∎

Example 4 In a simple modification of example 3 we consider both the possible removal or addition of bars. This is relevant if both missing as well as additional bars can potentially reduce the performance of the structure. Thus the deficiency sets of eq.(5) become:

$$\mathcal{D}(\alpha, \widetilde{p}) = \{p: \ p \in \mathcal{T}, \ |p - \widetilde{p}| = \min[\alpha, \ \dim(\widetilde{p})]\}, \quad \alpha = 0, 1, 2, \ldots \tag{6}$$

Note that, when the horizon of uncertainty equals or exceeds the dimension of the vectors p, then exactly $\dim(\widetilde{p})$ elements differ between p and $\widetilde{p}$. In other words, $\mathcal{D}(\alpha, \widetilde{p})$ remains constant for all $\alpha \geq \dim(\widetilde{p})$. In fact, this deficiency set will contain only one element: the binary complement of $\widetilde{p}$. ∎

Example 5 In continuation of example 2 let the deficiency set $\mathcal{D}(\alpha, \widetilde{p})$ be the set of all structures in $\mathcal{T}$ in which one cross section is extremal (either its nominal or its minimal value) and all others are within the range of values at that horizon of uncertainty. ∎

The **strong redundancy** of design $\widetilde{p}$ is the greatest deficiency which can be tolerated at any place in the structure, without violating the performance requirement, eq.(4). Whatever the specific form of the deficiency set, the strong redundancy is defined as:

$$\sigma(\widetilde{p}, \overline{g}) = \max \left\{ \alpha: \ \left(\max_{p \in \mathcal{D}(\alpha, \widetilde{p})} g(p) \right) \leq \overline{g} \right\} \tag{7}$$

We define $\sigma(\widetilde{p}, \overline{g}) = 0$ if the set of α values in eq.(7) is empty.

For notational convenience we sometimes denote $\sigma(\widetilde{p}, \overline{g})$ as σ. $\mathcal{D}(\sigma, \widetilde{p})$ is the deficiency set evaluated at the horizon of uncertainty which equals the strong deficiency. By definition of strong deficiency, all elements of $\mathcal{D}(\sigma, \widetilde{p})$ satisfy the performance requirement, eq.(4). Furthermore, for any $\alpha > \sigma$ there is at least one element of $\mathcal{D}(\alpha, \widetilde{p})$ which does not satisfy eq.(4). However, the definition of strong redundancy does not stipulate anything about how many elements of deficiency sets $\mathcal{D}(\alpha, \widetilde{p})$ satisfy eq.(4) for $\alpha < \sigma$. For instance, $\mathcal{D}(0, \widetilde{p})$ will usually be defined to contain only the nominal structure and it may or may not satisfy eq.(4). More generally, it can happen that the performance displays "non-monotonic" behavior: adding a bar can reduce the performance. We will define monotonicity later (definition 2), and present an example of non-monotonicity in section 4.4. In summary, the composition of $\mathcal{D}(\alpha, \widetilde{p})$ changes as α increases from 0 to σ. At any given $\alpha < \sigma$ the structures in $\mathcal{D}(\alpha, \widetilde{p})$ may or may not all satisfy eq.(4).

3 Info-Gap Uncertainty and Robustness

Let $\widetilde{p}$ denote the nominal design specification of the system. The true value, p, which represents the real physical structure, is uncertain due to wear, aging, negligence, manufacturing variability, etc. The uncertainty is represented by an info-gap model, $\mathcal{U}(\alpha, \widetilde{p})$, which is a family of nested sets of p values. There are many types of info-gap models (see Ben-Haim 2006), but all info-gap models have two properties. *Contraction* asserts that, in the absence of uncertainty, the nominal design is the only possibility:

$$\mathcal{U}(0, \widetilde{p}) = \{\widetilde{p}\} \tag{8}$$

Nesting asserts that the range of possible realizations of p increases as the horizon of uncertainty, α, increases:

$$\alpha < \alpha' \quad \text{implies} \quad \mathcal{U}(\alpha, \widetilde{p}) \subseteq \mathcal{U}(\alpha', \widetilde{p}) \tag{9}$$

We will assume that the sets of our info-gap models are closed. This is not necessary in principle, though it does not exclude any info-gap models which are important in practice, and it does simplify our proofs.

Example 6 In continuation of example 1 consider the following info-gap model for uncertainty. Let us consider a specific truss design, denoted $\widetilde{p}$. We are concerned about possible absences of bars from this design. (We are not worried about possible additional bars.) We don't think that any bars are missing, since $\widetilde{p}$ is the official (nominal) design, but an unknown number of bars *might* be missing due to damage or negligence, etc. In short, we consider uncertainty about the possible absence of bars. We define an info-gap model with a discrete horizon of uncertainty as follows. $\mathcal{U}(\alpha, \widetilde{p})$ is the set of all trusses in $\mathcal{T}$ from which no more than α bars are missing:

$$\mathcal{U}(\alpha, \widetilde{p}) = \{p : \ p \in \mathcal{T}, \ p \le \widetilde{p}, \ |p - \widetilde{p}| \le \alpha\}, \quad \alpha = 0, 1, 2, \ldots \tag{10}$$

∎

Example 7 In a simple modification of example 6 we consider both the possible removal or addition of bars. This is relevant if both missing as well as additional bars can potentially reduce the performance of the structure. Thus the info-gap model of eq.(10) becomes:

$$\mathcal{U}(\alpha, \widetilde{p}) = \{p : \ p \in \mathcal{T}, \ |p - \widetilde{p}| \le \alpha\}, \quad \alpha = 0, 1, 2, \ldots \tag{11}$$

∎

Example 8 In continuation of example 2, we are concerned about possible corrosion, abrasion or cracking of bars resulting in reduced cross-sectional area. The uncertainty is now continuous and represented in various ways, depending on the type of prior information available.

One info-gap model is a direct continuous extension of the info-gap model in eq.(10), the only difference being that now the true design vector p and the horizon of uncertainty α are continuous variables:

$$\mathcal{U}(\alpha, \widetilde{p}) = \{p : \ 0 \le p \le \widetilde{p}, \ |p - \widetilde{p}| \le \alpha\}, \quad \alpha \ge 0 \tag{12}$$

$\mathcal{U}(\alpha, \widetilde{p})$ now contains all trusses whose cumulative degradation with respect to the design, $\widetilde{p}$, is no greater than α which is continuous.

A slightly more subtle extension of the discrete info-gap model in eq.(10) is:

$$\mathcal{U}(\alpha, \widetilde{p}) = \left\{ p : \begin{array}{c} \sum_{i=1}^{N} t_i > N - \alpha - 1, \ t_i \in \{0, \ 1\} \\ 0 \le d_i \le 1 - t_i, \ \sum_{i=1}^{N} d_i \le \alpha \\ p_i = (1 - d_i)\widetilde{p}_i, \ i = 1, \ldots, N \end{array} \right\}, \quad \alpha \ge 0 \tag{13}$$

As an example, suppose $\alpha = 0.1$. The first row, $\sum_{i=1}^{N} t_i > N - 1.1$, means that at most one bar can have $t_i = 0$. Denote this bar by $i^\star$. All other bars have $t_i = 1$. The second row states $0 \le d_{i^\star} \le 0.1$ while all other bars have $d_i = 0$. The third row states that $0.9\widetilde{p}_{i^\star} \le p_{i^\star} \le \widetilde{p}_{i^\star}$ while all other bars have $p_i = \widetilde{p}_i$. When the horizon of uncertainty, α, is restricted to integer values, then the info-gap model of eq.(13) reverts to the info-gap model of eq.(10).

Note that, for arbitrary continuous α in eq.(13), the number of damaged bars increments discretely (1st row). However, among damaged bars the degree of damage is continuous (2nd and 3rd rows).

One can also formulate an info-gap model in which the uncertain degradation of each bar is specified separately. The following fractional-error info-gap model is commonly used:

$$\mathcal{U}(\alpha, \widetilde{p}) = \left\{ p : \ 0 \le p, \ 0 \le \frac{\widetilde{p}_i - p_i}{\widetilde{p}_i} \le \alpha \text{ for all } i \right\}, \quad \alpha \ge 0 \tag{14}$$

∎

Robustness. For any info-gap model of uncertainty, the robustness is the greatest horizon of uncertainty up to which the performance requirement, eq.(4), is satisfied:

$$\widehat{\alpha}(\widetilde{p}, \overline{g}) = \max \left\{ \alpha : \ \left(\max_{p \in \mathcal{U}(\alpha, \widetilde{p})} g(p) \right) \le \overline{g} \right\} \tag{15}$$

We define $\widehat{\alpha}(\widetilde{p}, \overline{g}) = 0$ if the set of α values in eq.(15) is empty.

Example 9 We evaluate the robustness function for the info-gap model of eq.(11). The horizon of uncertainty is discrete so the robustness also takes only non-negative integer values. We will assume that $\mathcal{T}$, the set of possible structures, includes all possible combinations of N bars where $N = 5$. Let the nominal structure be $\widetilde{p} = (1, 0, 1, 1, 0)$. The performance function is linear, $g(p) = \psi^T p$, where $\psi = (5, 3, 1, -2, -4)$. Adding a bar can either improve or reduce performance (lower or raise $g(p)$, respectively) because ψ has both negative and positive elements. This performance function is thus not monotonic in the sense to be defined subsequently in definition 2.

Let $\mu(\alpha)$ denote the inner maximum in the definition of the robustness, eq.(15). The robustness is the greatest integer value of α for which $\mu(\alpha) \leq \overline{g}$. We note that $\mu(\alpha)$ increases as α increases due to the nesting of the sets, $\mathcal{U}(\alpha, \widetilde{p})$, of the info-gap model. This means that the robustness is actually the greatest integer value of α for which $\mu(\alpha) \leq \overline{g}$. In other words, a plot of α vertically vs $\mu(\alpha)$ horizontally is the same as a plot of $\widehat{\alpha}(\widetilde{p}, \overline{g})$ vertically vs $\overline{g}$ horizontally. In short, $\mu(\alpha)$ is the inverse of $\widehat{\alpha}(\widetilde{p}, \overline{g})$ at fixed $\widetilde{p}$. We will derive an expression for $\mu(\alpha)$.

When $\alpha = 0$ we can only choose $p = \widetilde{p}$ so $\mu(0) = 5 + 1 + (-2) = 4$.

When $\alpha = 1$ we can choose at most one element of p to differ from $\widetilde{p}$. We maximize $\psi^T p$ by choosing $p = (1, 1, 1, 1, 0)$ so $\mu(1) = 5 + 3 + 1 + (-2) = 7$.

When $\alpha = 2$ we can choose at most two elements of p to differ from $\widetilde{p}$. We maximize $\psi^T p$ by choosing $p = (1, 1, 1, 0, 0)$ so $\mu(2) = 5 + 3 + 1 = 9$.

When $\alpha > 2$ we choose the same structure, $p = (1, 1, 1, 0, 0)$, so $\mu(\alpha) = 9$.

The robustness function is obtained by plotting α vertically vs $\mu(\alpha)$ horizontally as shown in fig. 1. The robustness curve thus has the following structure, as shown by the heavy line in fig. 1:

$\widehat{\alpha}(\widetilde{p}, \overline{g}) = 0$ for $\overline{g}$ in $(-\infty, 7)$.

$\widehat{\alpha}(\widetilde{p}, \overline{g}) = 1$ for for $\overline{g}$ in $[7, 9)$.

$\widehat{\alpha}(\widetilde{p}, \overline{g}) = \infty$ for $\overline{g}$ in $[9, \infty)$. ∎

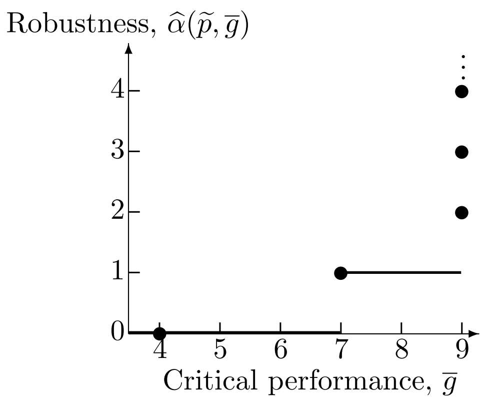

Figure 1: Robustness curve for example 9.

4 Redundancy and Robustness

The intuitive connection between redundancy and robustness to uncertainty is fairly obvious, as suggested in section 1. In this section we establish two propositions which quantify this relation and reveal some of its subtleties.

4.1 Strong Redundancy is Bounded by Robustness

Comparing the definitions of strong redundancy and robustness in eqs.(7) and (15) we see a marked formal similarity. The following proposition establishes a relation between strong redundancy and robustness. Later propositions will establish stronger relations, which depend on stronger assumptions.

Proposition 1 *If the deficiency set belongs to the info-gap model then the strong redundancy is never less than the robustness.*
If:

$$\mathcal{D}(\alpha, \widetilde{p}) \subseteq \mathcal{U}(\alpha, \widetilde{p}) \quad \textit{for all } \alpha \geq 0 \tag{16}$$

Then:

$$\sigma(\widetilde{p}, \overline{g}) \geq \widehat{\alpha}(\widetilde{p}, \overline{g}) \tag{17}$$

All proofs appear in section 9.

Example 10 We now illustrate the strong redundancy for the deficiency sets $\mathcal{D}(\alpha, \widetilde{p})$ in eq.(6). We will use the performance function and nominal structure introduced in example 9. Let $\phi(\alpha)$ denote the inner maximum in the definition of the strong redundancy, eq.(7). Thus $\phi(\alpha)$ is the greatest value of $\psi^T p$ when *exactly* α bars are changed (either added or missing) from the nominal structure. We proceed as in example 9, though an important difference will emerge.

When $\alpha = 0$ we can only choose $p = \widetilde{p}$ so $\phi(0) = 5 + 1 + (-2) = 4$ which equals $\mu(0)$.

When $\alpha = 1$ we must choose exactly one element of p to differ from $\widetilde{p}$. We maximize $\psi^T p$ by choosing $p = (1, 1, 1, 1, 0)$ so $\phi(1) = 5 + 3 + 1 + (-2) = 7$, which equals $\mu(1)$.

When $\alpha = 2$ we must choose exactly two elements of p to differ from $\widetilde{p}$. We maximize $\psi^T p$ by choosing $p = (1, 1, 1, 0, 0)$ so $\phi(2) = 5 + 3 + 1 = 9$, which equals $\mu(2)$.

When $\alpha = 3$ we must choose exactly three elements of p to differ from $\widetilde{p}$. We maximize $\psi^T p$ by choosing $p = (1, 1, 0, 0, 0)$ so $\phi(3) = 5 + 3 = 8$, which is less than $\mu(3)$.

When $\alpha = 4$ we must choose exactly four elements of p to differ from $\widetilde{p}$. We maximize $\psi^T p$ by choosing $p = (1, 1, 0, 0, 1)$ so $\phi(4) = 5 + 3 + (-4) = 4$, which is less than $\mu(4)$.

When $\alpha = 5$ we must choose exactly five elements of p to differ from $\widetilde{p}$. We maximize $\psi^T p$ by choosing $p = (0, 1, 0, 0, 1)$ so $\phi(4) = 3 + (-4) = -1$, which is less than $\mu(5)$.

When $\alpha > 5$ we choose the same structure, $p = (0, 1, 0, 0, 1)$, so $\phi(\alpha) = -1$.

We notice that, in all cases, $\phi(\alpha) \leq \mu(\alpha)$. This is directly related to the inequality which is asserted by proposition 1.

What does the strong redundancy function, $\sigma(\widetilde{p}, \overline{g})$, look like, as a function of the critical performance, $\overline{g}$? The strong redundancy at $\overline{g}$ is the greatest value of α at which $\phi(\alpha) \leq \overline{g}$. Hence, from the above values of $\phi(\alpha)$, we see that:

$\sigma(\widetilde{p}, \overline{g}) = 0$ for $\overline{g} < -1$, because $\phi(\alpha) < -1$ for no value of α so the set of α values in eq.(7) is empty.

$\sigma(\widetilde{p}, \overline{g}) = \infty$ for $\overline{g} \geq -1$, because $\phi(\alpha) = -1$ for all $\alpha \geq 5$, so the maximum such α is infinite.

Comparing this with the robustness function in example 9 we find:

$$\sigma(\widetilde{p}, \overline{g}) = \widehat{\alpha}(\widetilde{p}, \overline{g}) \quad \text{for } \overline{g} < -1 \tag{18}$$

$$\sigma(\widetilde{p}, \overline{g}) > \widehat{\alpha}(\widetilde{p}, \overline{g}) \quad \text{for } -1 \leq \overline{g} < 9 \tag{19}$$

$$\sigma(\widetilde{p}, \overline{g}) = \widehat{\alpha}(\widetilde{p}, \overline{g}) \quad \text{for } 9 \leq \overline{g} \tag{20}$$

This of course is consistent with proposition 1. ▮

4.2 Coherence, Monotonicity and Unsaturation

In this section we define three central concepts in preparation for the equivalence theorem in section 4.3.

In many cases the deficiency set $\mathcal{D}(\alpha, \widetilde{p})$ is the "outer layer" or boundary of $\mathcal{U}(\alpha, \widetilde{p})$, as for example eqs.(5) and (10). This leads us to define the following common property of the info-gap model and the deficiency set of a system.

Definition 1 *The info-gap model, $\mathcal{U}(\alpha, \widetilde{p})$, and the deficiency set, $\mathcal{D}(\alpha, \widetilde{p})$, of a system are* coherent *if:*

$$\mathcal{U}(\alpha, \widetilde{p}) = \bigcup_{x=0}^{\alpha} \mathcal{D}(x, \widetilde{p}) \tag{21}$$

This definition applies to both discrete and continuous horizon of uncertainty, α. The union in eq.(21) must be interpreted accordingly.

Definition 2 *The performance function* $g(p)$ *is* monotonic *in the uncertain vector* p *if* $g(p)$ *improves (gets smaller) as* p *gets larger:*

$$p' \le p \quad \textit{implies} \quad g(p') \ge g(p) \tag{22}$$

Definition 3 *An info-gap model* $\mathcal{U}(\alpha, \widetilde{p})$ does not saturate *up to horizon of uncertainty* $\alpha_{\max}$ *if, for all* $\alpha < \alpha'$ *where* $\alpha' \le \alpha_{\max}$*, and for each* $p \in \mathcal{U}(\alpha, \widetilde{p})$*, there is a* $p' \in \mathcal{U}(\alpha', \widetilde{p}) - \mathcal{U}(\alpha, \widetilde{p})$ *such that* $p' \le p$*.*

Non-saturation of an info-gap model means that, as the horizon of uncertainty increases (up to some limiting value $\alpha_{\max}$), new structures are introduced which are no stronger (in fact, usually weaker) than previously included structures. This means, approximately, that more uncertainty always entails the possibility of worse (weaker) structures, up to some limiting horizon of uncertainty. Most of the info-gap models encountered in this paper do saturate at some finite horizon of uncertainty, usually when all physically meaningful or possible structural decrements are included. Larger horizons of uncertainty are not of interest (usually, no additional structures exist).

Finally, let us define the set of all maximizing p's in a specified info-gap model at horizon of uncertainty α:

$$P(\alpha) = \left\{ p \in \mathcal{U}(\alpha, \widetilde{p}) : \ g(p) = \max_{p' \in \mathcal{U}(\alpha, \widetilde{p})} g(p') \right\} \tag{23}$$

4.3 An Equivalence Theorem

The info-gap robustness, $\widehat{\alpha}(\widetilde{p}, \overline{g})$ in eq.(15), is the greatest horizon of uncertainty in the structure up to which the performance requirement, eq.(4), is guaranteed to be satisfied. The strong redundancy, $\sigma(\widetilde{p}, \overline{g})$ in eq.(7), is the greatest decrement in structural integrity which can be tolerated without violating the performance requirement. The following proposition states that, if the performance function is monotonic, if the info-gap model does not saturate, and if the deficiency sets are coherent with the info-gap model, then robustness and strong redundancy are equivalent.

Proposition 2 *Strong redundancy, eq.(7), and info-gap robustness, eq.(15), are equivalent if monotonicity, coherence and non-saturation hold.*
Given:
- $g(p)$ *is monotonic.*
- $\mathcal{U}(\alpha, \widetilde{p})$ *and* $\mathcal{D}(\alpha, \widetilde{p})$ *are coherent.*
- $\mathcal{U}(\alpha, \widetilde{p})$ *does not saturate up to horizon of uncertainty* $\alpha_{\max}$.
- $P(\alpha)$ *is non-empty for all* $\alpha \ge 0$.
- *And* $\overline{g}$ *is such that:*

$$\widehat{\alpha}(\widetilde{p}, \overline{g}) \le \alpha_{\max} \quad \textit{and} \quad \sigma(\widetilde{p}, \overline{g}) \le \alpha_{\max} \tag{24}$$

Then:

$$\widehat{\alpha}(\widetilde{p}, \overline{g}) = \sigma(\widetilde{p}, \overline{g}) \tag{25}$$

Proposition 1 establishes that the strong redundancy is bounded below by the robustness, with almost no assumptions. Proposition 2, in contrast, establishes conditions—still quite general—under which the strong redundancy and the robustness are identical.

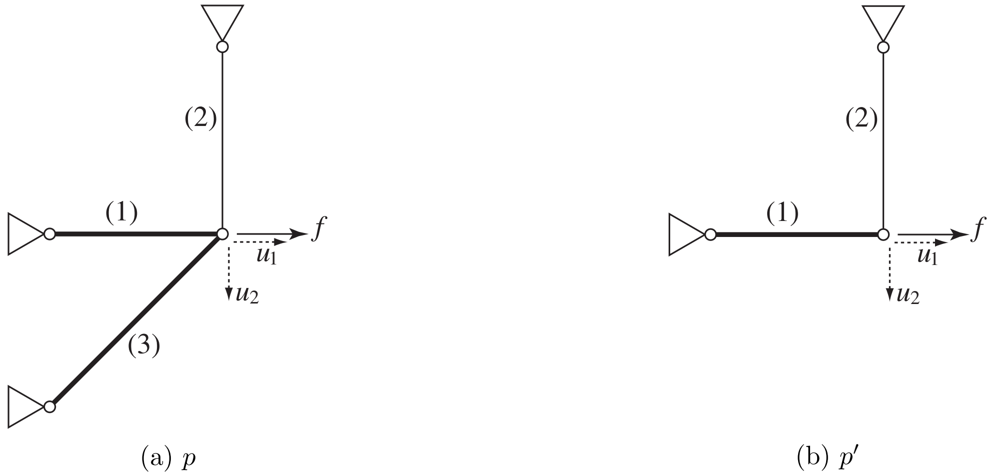

Figure 2: An example in which the performance function $g(p)$ is not monotonic with respect to p.

Table 1: The vector of the design variables p and the corresponding displacement vector u of the truss in fig. 2.

	p	p'
member (1)	10.0	10.0
member (2)	0.1	0.1
member (3)	10.0	0.0
u_1	0.09904	0.10000
u_2	0.09631	0.00000

4.4 Example of Non-Monotonic Performance

The equivalence between strong redundancy and robustness, proposition 2, depends on the performance function being monotonic, definition 2. Examples 9 and 10 were a hypothetical illustration of a non-monotonic performance function. We now show a simple mechanical structure which violates monotonicity.

Consider the 3-bar truss in fig. 2(a). A horizontal external load $f = 1.0$ is applied at the free node. Young's modulus is 1.0, and the lengths of the horizontal and vertical members are 1.0. The vector of the member cross-sectional areas, given in table 1, is the design vector p.

The vertical displacement of the free node is the performance function $g(p)$:

$$g(p) = u_2. \tag{26}$$

We define the design p' by removing the member (3) from p, as illustrated in fig. 2(b). The vertical displacement in structure p' is less than in p, as seen in table 1. That is, the performance function for this class of structures is not monotonic according to definition 2 because a *decrement* of the structure results in an *improvement* in the performance function:

$$p' \leq p, \quad g(p') < g(p). \tag{27}$$

Thus the performance function in this example—the displacement constraint—does not satisfy the monotonicity in definition 2. Since the member stress is a linear function of the displacement, the stress constraint is also non-monotonic in this example.

This example demonstrates that the monotonicity which is posited in proposition 2 does not hold for all structures or all performance functions.

5 Example: Strong Redundancy with Stability Constraint

(a) Nominal truss A (b) Nominal truss B

Figure 3: Two nominal truss designs with $N = 25$.

(a) truss A (b) truss B

Figure 4: Examples of $p \in \mathcal{U}(\alpha, \widetilde{p})$ which violates the stability constraint for the trusses in fig. 3.

In this example we consider the strong redundancy of a truss concerning the stability constraint treated in example 1. The deficiency of bars from the nominal designs is defined according to example 3.

Consider the two nominal truss designs shown in fig. 3, where these trusses consist of $N = 25$ bars and a common collection of nodes. In the deficiency model in example 3, $p \in \mathcal{D}(\alpha, \widetilde{p})$ means that p corresponds to a situation in which exactly α bars are missing from $\widetilde{p}$ in fig. 3. The performance requirement in eq.(4) is defined as the stability constraint. Specifically, the strong redundancy $\sigma(\widetilde{p}, \overline{g})$ is the maximal number of bars such that the truss is still stable if any set of $\sigma(\widetilde{p}, \overline{g})$ bars is missing.

The nominal truss A becomes unstable if a particular bar is removed, as shown in fig. 4(a), while after removing any single bar from nominal truss B it is still stable. From this observation, it is intuitively natural to consider the truss B as being more redundant than the truss A. Note that those two trusses share the same degree of static indeterminacy $s = 9$. The strong redundancy of truss A is $\sigma(\widetilde{p}_{\mathrm{A}}, \overline{g}) = 0$, while that of the truss B is $\sigma(\widetilde{p}_{\mathrm{B}}, \overline{g}) = 2$. Indeed, if a particular set of three bars is missing, then the truss B becomes unstable, as shown in fig. 4(b). Thus the value of the strong redundancy function is consistent with the intuitive notion of redundancy.

6 Example: Strong Redundancy with Compliance Constraint

(a) Nominal truss A (b) Nominal truss B

Figure 5: Two nominal truss designs for the example of compliance constraint, section 6.

We consider the strong redundancy of a truss associated with the compliance constraint introduced in example 2. The deficiency of bars from the nominal designs is defined according to $\mathcal{D}(\alpha, \widetilde{p})$ introduced in example 3.

Consider the two nominal truss designs shown in fig. 5, where the trusses A and B consist of $N = 30$ bars and $N = 26$ bars, respectively. Suppose that a downward vertical load of 1.0 and a rightward horizontal load of 0.5 are applied to the lower rightmost node. Young's modulus is 1.0, and the lengths of the horizontal and vertical members are 1.0. We consider the compliance constraint as

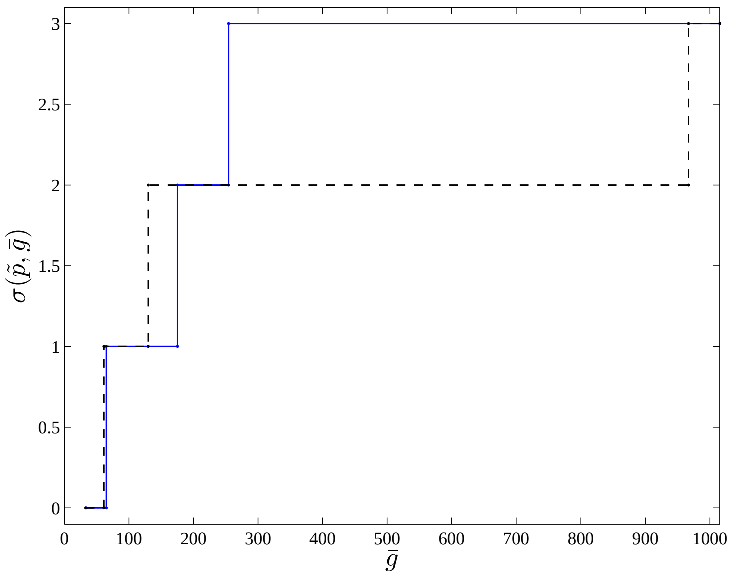

Figure 6: The variation of $\sigma(\widetilde{p},\bar{g})$ with respect to $\bar{g}$ for the compliance constraint. '——': Truss A; '– – –': Truss B.

the performance requirement, i.e., $g(p)$ in eq.(4) denotes the compliance, while $\bar{g}$ is the upper bound for the compliance.

Suppose that the cross-sectional area of each undeficient member is 1.0 for both truss A and truss B. The deficiency set $\mathcal{D}(\alpha,\widetilde{p})$ is defined by eq.(5). The variation of the strong redundancy $\sigma(\widetilde{p},\bar{g})$ with respect to $\bar{g}$ is depicted in fig. 6. It is observed that these redundancy curves cross at $\sigma(\widetilde{p},\bar{g}) = 2$. This has important design implications as we shall see.

The nominal preference between the two structures, based on their estimated compliances, is B $\succ$ A, because B has lower estimated compliance than A: $33.0 = g(\widetilde{p}_{\rm B}) < g(\widetilde{p}_{\rm A}) = 33.8$. However, the redundancies for the estimated compliances are precisely zero. That is:

$$\sigma[{\rm A}, g(\widetilde{p}_{\rm A})] = 0 = \sigma[{\rm B}, g(\widetilde{p}_{\rm B})] \tag{28}$$

In other words, aspiring to compliance as small as $g(\widetilde{p}_{\rm B})$ for truss B is unreliable in the sense that the truss has no structural redundancy for supplying this compliance. The situation is the same regarding truss A.

It is thus important to consider the value of the redundancy function when assessing a truss with respect to any specified complicance requirement. More redundancy is preferable to less redundancy, so—when considering their redundancy—one would prefer design $\widetilde{p}$ over design $\widetilde{p}'$ if the former has more redundancy:

$$\widetilde{p} \succ \widetilde{p}' \quad \text{if} \quad \sigma(\widetilde{p},\bar{g}) > \sigma(\widetilde{p}',\bar{g}) \tag{29}$$

We note however that the inequality on the right may depend on the required compliance, $\bar{g}$. In other words, the redundancy-preference between the designs may change as the required compliance changes.

This is precisely what happens in the current example. From fig. 6 we see that design B is preferred over design A, according to the preference relation in eq.(29), for the following values of $\bar{g}$:

$$\text{B} \succ \text{A} \quad \text{for} \quad \bar{g} \in (61,\ 65) \cup (130,\ 175) \tag{30}$$

However, the redundancy preferences are reversed for a higher range of $\bar{g}$ values:

$$\text{A} \succ \text{B} \quad \text{for} \quad \bar{g} \in (254,\ 967) \tag{31}$$

For all other values of critical compliance, $\bar{g}$, the redundancies of the two structures are the same, and the designer would be indifferent between then, as far as redundancy is concerned.

7 Weak Redundancy

The strong redundancy defined in eq.(7) is the greatest horizon of uncertainty, α, at which *all* structures in the deficiency set $\mathcal{D}(\alpha, \widetilde{p})$ satisfy the performance requirement, eq.(4).

We will now define a concept of weak redundancy and show its relationship to strong redundancy. The *weak redundancy of order n* is the greatest horizon of uncertainty at which *at least n* structures in the deficiency set satisfy the performance requirement, eq.(4). Formally:

$$\sigma_n(\widetilde{p}, \overline{g}) = \max\left\{\alpha : \ g(p) \le \overline{g} \ \text{ for at least } n \text{ elements of } \mathcal{D}(\alpha, \widetilde{p})\right\} \tag{32}$$

For notational convenience we sometimes denote $\sigma_n(\widetilde{p}, \overline{g})$ as σ_n. By definition, at least n elements of $\mathcal{D}(\sigma_n, \widetilde{p})$ satisfy eq.(4). Furthermore, only fewer than n elements of $\mathcal{D}(\alpha, \widetilde{p})$ satisfy eq.(4) for all $\alpha > \sigma_n$. However, the definition of weak redundancy does not stipulate anything about how many elements of deficiency sets $\mathcal{D}(\alpha, \widetilde{p})$ satisfy eq.(4) for $\alpha < \sigma_n$. For instance, no more than one element of $\mathcal{D}(0, \widetilde{p})$ can satisfy eq.(4) because $\mathcal{D}(0, \widetilde{p})$ is defined to contain only the nominal structure. If α is discrete then the size of $\mathcal{D}(\alpha, \widetilde{p})$, denoted $|\mathcal{D}(\alpha, \widetilde{p})|$, is also an integer and may vary up and down as α increases, depending on how the deficiency sets are defined. If α is continuous then $|\mathcal{D}(\alpha, \widetilde{p})|$ will be infinite for all $\alpha > 0$. For both discrete and continuous α, the composition of $\mathcal{D}(\alpha, \widetilde{p})$ changes as α increases from 0 to σ. The number of structures in $\mathcal{D}(\alpha, \widetilde{p})$ which satisfy eq.(4) may vary up and down as α increases between 0 and σ_n.

The following proposition establishes elementary relations between weak and strong redundancy. We first define two parameters:

$$\widehat{n}_\sigma = |\mathcal{D}\left(\sigma(\widetilde{p}, \overline{g}), \widetilde{p}\right)| \tag{33}$$

$$\widehat{n}_\alpha = |\mathcal{D}\left(\widehat{\alpha}(\widetilde{p}, \overline{g}), \widetilde{p}\right)| \tag{34}$$

$\widehat{n}_\sigma$ is the size of the deficiency set evaluated at the strong redundancy. $\widehat{n}_\alpha$ is the size of the deficiency set evaluated at the robustness. If the horizon of uncertainty, α, is discrete then $\widehat{n}_\sigma$ and $\widehat{n}_\alpha$ are finite. If the horizon of uncertainty is continuous and if $\sigma(\widetilde{p}, \overline{g}) > 0$ then $\widehat{n}_\sigma$ is infinite. Likewise, if the horizon of uncertainty is continuous and if $\widehat{\alpha}(\widetilde{p}, \overline{g}) > 0$ then $\widehat{n}_\alpha$ is infinite.

The weak redundancy is the quantification of a different intuition about redundancy than the intuition which underlies the strong redundancy. Proposition 3 demonstrates some basic relations between these concepts. In addition, it plays a role in the proof of proposition 4.

Proposition 3 *The weak redundancy decreases with increasing order and, for continuous α, is never less than the strong redundancy and never less than the robustness if the deficiency set is included in the info-gap model.*
It is asserted *that, for any finite n:*

$$\sigma_n(\widetilde{p}, \overline{g}) \ \ge \ \sigma_{n+1}(\widetilde{p}, \overline{g}) \tag{35}$$

$$\sigma_{\widehat{n}_\sigma}(\widetilde{p}, \overline{g}) \ \ge \ \sigma(\widetilde{p}, \overline{g}) \tag{36}$$

and, *for continuous α and any finite n:*

$$\sigma_n(\widetilde{p}, \overline{g}) \ \ge \ \sigma(\widetilde{p}, \overline{g}) \tag{37}$$

Also, if:

$$\mathcal{D}(\alpha, \widetilde{p}) \subseteq \mathcal{U}(\alpha, \widetilde{p}) \quad \textit{for all } \alpha \ge 0 \tag{38}$$

Then:

$$\sigma_{\widehat{n}_\alpha}(\widetilde{p}, \overline{g}) \ \ge \ \widehat{\alpha}(\widetilde{p}, \overline{g}) \tag{39}$$

and, *for continuous α and any finite n:*

$$\sigma_n(\widetilde{p}, \overline{g}) \ \ge \ \widehat{\alpha}(\widetilde{p}, \overline{g}) \tag{40}$$

Examination of the proofs of eqs.(35)—(37) in section 9 shows that these equations do not depend on the supposition in eq.(38).

Our final proposition is motivated by the following question: Will removal of an element from the structure reduce the strong redundancy? More precisely, what is a sufficient condition, under which, removal of an element will reduce the strong redundancy? The following proposition shows, for a specific type of deficiency set, that a sufficient condition is equality between the strong redundancy and the weak redundancy of order 1.

As in example 1 and elsewhere, we again use p as a binary indicator vector where each element p_i is either 0 or 1 to indicate absence of presence, respectively, of the ith structural component. However, algebraic operations on binary vectors are defined in the usual real vector field $\Re^N$. Thus, when p and p' are binary vectors, the elements of $p - p'$ take the values -1, 0 or 1. Finally, the vector norm used in the following proposition is:

$$|p| = \sum_{i=1}^{N} |p_i|. \tag{41}$$

Proposition 4 *Removal of an element reduces the strong redundancy if the strong redundancy equals the weak redundancy of order 1.*
Given:

- *$g(p)$ is monotonic.*
- *The deficiency set $\mathcal{D}(\alpha, \widetilde{p})$ is defined by eq.(5).*
- *The following equality holds:*

$$\sigma_1(\widetilde{p}, \bar{g}) = \sigma(\widetilde{p}, \bar{g}) \tag{42}$$

Then:

$$\sigma(\widetilde{p}', \bar{g}) < \sigma(\widetilde{p}, \bar{g}) \quad (\forall \widetilde{p}' : \widetilde{p}' \leq \widetilde{p},\ \widetilde{p}' \neq \widetilde{p}). \tag{43}$$

The following example demonstrates a simple structure which satisfies the condition of eq.(42) in proposition 4.

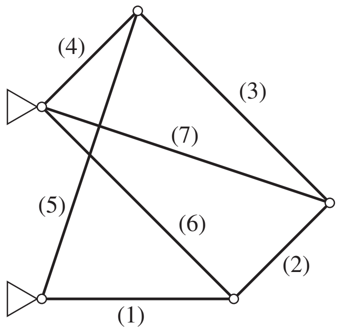

Figure 7: A truss satisfying the condition in Proposition 4 with respect to the stability constraint.

Example 11 Consider the truss illustrated in fig. 7. Members (2) and (4) are parallel, and members (3) and (6) are parallel. We consider the stability constraint introduced in Example 1.

The truss in fig. 7 is still stable after removing any single member. However, the truss becomes unstable by removing members (1) and (2). Therefore, $\sigma(\widetilde{p}, \bar{g}) = 1$. Indeed, the truss becomes unstable after removing any set of two members. Therefore, $\sigma_1(\widetilde{p}, \bar{g}) = 1$. ∎

8 Discussion

Summary. Redundancy is a measure of the amount of damage that a structure can sustain without losing some specified elements of its functionality. Robustness is a measure of the amount of environmental uncertainty against which a structure will be able to maintain specified functionality. These ideas are opposite sides of the same coin. Roughly speaking, redundancy measures the structure while robustness measures the environment. Obviously, the two concepts are intimately connected, though subtleties are revealed when one quantifies the relationship. In this paper we have defined two concepts of redundancy—strong, eq.(7), and weak, eq.(32)—and a concept of robustness, eq.(15), and we have explored the relations between them.

Proposition 1 establishes that, if all structural deficiencies against which the redundancy is evaluated are also environmental contingencies against which the robustness is evaluated, then the robustness is a lower bound for the strong redundancy. Significantly, the strong redundancy can actually exceed the robustness as demonstrated by examples 9 and 10. The conditions of proposition 1 are not sufficient to make strong redundancy and robustness equivalent; they each reveal a somewhat different aspect of structural durability.

Proposition 2, however, establishes quite general conditions—stronger than those of proposition 1—in which the strong redundancy and the robustness are in fact numerically identical. When these conditions hold, one simply does not need both functions; either one of them is redundant. Propositions 1 and 2 quantify and delimit the intuition that redundancy and robustness are intimately connected.

However, even though the conditions of proposition 2 apply to a wide range of mechanical structures, the example in section 4.4 shows a simple truss which violates the conditions of proposition 2. For these situations the strong redundancy and the robustness reveal different aspects of the structure.

Even when the strong redundancy and the robustness are equivalent, they still do not reflect all aspects of survivability after structural degradation. One can define redundancy in different ways to reflect different aspects of these phenomena, even while keeping to a generic format applicable to virtually any mechanical system and any type of functionality. We have defined a weak redundancy in eq.(32) and explored its relation to strong redundancy and robustness. Proposition 3 establishes ranking relations among the three functions—robustness and strong and weak redundancy—which hold under very general conditions. Proposition 4 establishes, for an important class of applications, a sufficient condition (involving both strong and weak redundancy) in which any degradation necessarily reduces the strong redundancy.

Future directions. Our discussion of robustness and strong and weak redundancy suggests that no single measure of redundancy reflects all the subtleties of survival in uncertain environments. This paper has developed a generic framework for quantifying the intuitions of redundancy and robustness and for exploring the relations between these quantifications. Much remains to be explored. Other concepts of redundancy can be formulated to reflect other aspects of the processes of degradation and survival. For instance, we have not considered dynamic problems, though we believe that the concepts here are applicable to such situations. Also, the design implications of robustness and strong and weak redundancy can be explored, both on an abstract methodological level and for specific classes of structures.

9 Proofs

Proof of proposition 1. For notational convenience we denote $\sigma(\widetilde{p}, \overline{g})$ as σ, and we denote $\widehat{\alpha}(\widetilde{p}, \overline{g})$ as $\widehat{\alpha}$.

By the definition of robustness, all elements of $\mathcal{U}(\widehat{\alpha}, \widetilde{p})$ satisfy eq.(4). By the inclusion in eq.(16) all the elements of $\mathcal{D}(\widehat{\alpha}, \widetilde{p})$ satisfy eq.(4). Hence, by the definition of strong redundancy, $\sigma \geq \widehat{\alpha}$ which is eq.(17). ∎

The following two results, lemma 1 and 2, are needed in the proof of proposition 2.

Lemma 1 *If $g(p)$ is montonic and if $\mathcal{U}(\alpha, \widetilde{p})$ does not saturate up to horizon of uncertainty $\alpha_{\max}$, then any increase in horizon of uncertainty introduces new maximizing structures p.*
Given:

- *$g(p)$ is monotonic.*

• $\mathcal{U}(\alpha, \widetilde{p})$ *does not saturate up to horizon of uncertainty* $\alpha_{\max}$.
• $P(\alpha)$ *is non-empty for all* $\alpha \geq 0$.

Then, *for all* $\alpha \leq \alpha_{\max}$*:*

$$\alpha < \alpha' \quad \textit{implies} \quad P(\alpha') - P(\alpha) \neq \emptyset \tag{44}$$

Proof of lemma 1. We prove the lemma for continuous α. An analogous proof holds for discrete α.

(1) Let p be any element in $P(\alpha)$. By non-saturation, there exists a $p' \in \mathcal{U}(\alpha', \widetilde{p}) - \mathcal{U}(\alpha, \widetilde{p})$ such that:

$$p' \leq p \tag{45}$$

By monotonicity, this implies:

$$g(p') \geq g(p) \tag{46}$$

(2) Suppose this $p' \in P(\alpha')$. Then eq.(44) results because $p' \notin \mathcal{U}(\alpha, \widetilde{p})$ by non-saturation.

(3) Suppose this $p' \notin P(\alpha')$. Then, because $P(\alpha')$ is not empty, there exists $p^\star \in P(\alpha')$ such that:

$$g(p^\star) > g(p') \tag{47}$$

But eq.(46) implies:

$$g(p^\star) > g(p) \tag{48}$$

Thus $p^\star \notin \mathcal{U}(\alpha, \widetilde{p})$ which implies eq.(44). ∎

Lemma 2 *If* $g(p)$ *is montonic, if* $\mathcal{D}(\alpha, \widetilde{p})$ *and* $\mathcal{U}(\alpha, \widetilde{p})$ *are coherent, and if uncertainty does not saturate up to horizon of uncertainty* $\alpha_{\max}$*, then the maximum of* $g(p)$ *on* $\mathcal{U}(\alpha, \widetilde{p})$ *equals the maximum of* $g(p)$ *on* $\mathcal{D}(\alpha, \widetilde{p})$.

Given:
• $g(p)$ *is monotonic.*
• $\mathcal{U}(\alpha, \widetilde{p})$ *and* $\mathcal{D}(\alpha, \widetilde{p})$ *are coherent.*
• $\mathcal{U}(\alpha, \widetilde{p})$ *does not saturate up to horizon of uncertainty* $\alpha_{\max}$.
• $P(\alpha)$ *is non-empty for all* $\alpha \geq 0$.

Then, *for* $\alpha \leq \alpha_{\max}$*:*

$$\max_{p \in \mathcal{U}(\alpha, \widetilde{p})} g(p) = \max_{p \in \mathcal{D}(\alpha, \widetilde{p})} g(p) \tag{49}$$

Proof of lemma 2. We prove the lemma for continuous α. An analogous proof holds for discrete α.

By coherence, $\mathcal{U}(0, \widetilde{p}) = \mathcal{D}(0, \widetilde{p})$, which proves eq.(49) for $\alpha = 0$. We now need only consider $0 < \alpha \leq \alpha_{\max}$.

Coherence implies:

$$\mathcal{U}(\alpha, \widetilde{p}) = \mathcal{U}(\alpha - \mathrm{d}\alpha, \widetilde{p}) \cup \mathcal{D}(\alpha, \widetilde{p}) \tag{50}$$

Lemma 1 assures the existence of a structure p' such that:

$$p' \in P(\alpha) - P(\alpha - \mathrm{d}\alpha) \tag{51}$$

Clearly:

$$p' \notin \mathcal{U}(\alpha - \mathrm{d}\alpha, \widetilde{p}) \quad \text{and} \quad p' \in \mathcal{U}(\alpha, \widetilde{p}) \tag{52}$$

Thus, by coherence, eq.(50):

$$p' \in \mathcal{D}(\alpha, \widetilde{p}) \tag{53}$$

which implies eq.(49). ∎

Proof of proposition 2. Let σ denote $\sigma(\widetilde{p}, \overline{g})$ and let $\widehat{\alpha}$ denote $\widehat{\alpha}(\widetilde{p}, \overline{g})$.

(1) From the definition of strong redundancy in eq.(7) we see that:

$$\left(\max_{p \in \mathcal{D}(\sigma, \widetilde{p})} g(p) \right) \leq \overline{g} \tag{54}$$

This and lemma 2 imply:

$$\left(\max_{p \in \mathcal{U}(\sigma, \widetilde{p})} g(p) \right) \leq \overline{g} \tag{55}$$

From the definition of robustness in eq.(15) we conclude that:

$$\widehat{\alpha}(\widetilde{p}, \overline{g}) \geq \sigma(\widetilde{p}, \overline{g}) \tag{56}$$

(2) From the definition of robustness in eq.(15) we see that:

$$\left(\max_{p \in \mathcal{U}(\widehat{\alpha}, \widetilde{p})} g(p) \right) \leq \overline{g} \tag{57}$$

This and lemma 2 imply:

$$\left(\max_{p \in \mathcal{D}(\widehat{\alpha}, \widetilde{p})} g(p) \right) \leq \overline{g} \tag{58}$$

From the definition of strong redundancy in eq.(7) we conclude that:

$$\sigma(\widetilde{p}, \overline{g}) \geq \widehat{\alpha}(\widetilde{p}, \overline{g}) \tag{59}$$

(3) Eqs.(56) and (59) imply eq.(25). ∎

Proof of proposition 3. The proof is valid for both discrete and continuous α except regarding eqs.(37) and (40) treated in items (3) and (5) below. For notational convenience we denote $\sigma_n(\widetilde{p}, \overline{g})$ as σ_n, we denote $\sigma(\widetilde{p}, \overline{g})$ as σ, and we denote $\widehat{\alpha}(\widetilde{p}, \overline{g})$ as $\widehat{\alpha}$.

(1) To prove eq.(35): By definition of weak redundancy of order $n+1$, there are at least $n+1$ elements of $\mathcal{D}(\sigma_{n+1}, \widetilde{p})$ which satisfy eq.(4). Hence there are at least n elements of $\mathcal{D}(\sigma_{n+1}, \widetilde{p})$ which satisfy eq.(4). This implies that $\sigma_n \geq \sigma_{n+1}$ which is eq.(35).

(2) To prove eq.(36): By the definition of strong redundancy, all of the $\widehat{n}_\sigma$ elements of $\mathcal{D}(\sigma, \widetilde{p})$ satisfy eq.(4). Hence at least $\widehat{n}_\sigma$ elements of $\mathcal{D}(\sigma, \widetilde{p})$ satisfy eq.(4). Thus $\sigma_{\widehat{n}_\sigma}$ is at least as large as σ, which is eq.(36).

(3) To prove eq.(37): $\widehat{n}_\sigma = \infty$ for continuous α so eq.(35) implies that $\sigma_n \geq \sigma_{\widehat{n}_\sigma}$ for finite n. This with eq.(36) imply (37).

(4) To prove eq.(39): $\sigma \geq \widehat{\alpha}$ by proposition 1. This and eq.(36) imply eq.(39).

(5) To prove eq.(40): $\widehat{n}_\alpha = \infty$ for continuous α so eq.(35) implies that $\sigma_n \geq \sigma_{\widehat{n}_\alpha}$ for finite n. This with eq.(39) imply (40). ∎

Proof of proposition 4. We will denote $\sigma(\widetilde{p}, \overline{g})$ by σ. We will argue by contraposition, that is, it is sufficient to prove:

$$\exists \widetilde{p}' \leq \widetilde{p} \ (\widetilde{p}' \neq \widetilde{p}) : \sigma(\widetilde{p}', \bar{g}) \geq \sigma(\widetilde{p}, \bar{g}) \quad \Rightarrow \quad \sigma_1(\widetilde{p}, \bar{g}) > \sigma(\widetilde{p}, \bar{g}) \tag{60}$$

where, from proposition 3, we know that $\sigma_1(\widetilde{p}, \bar{g}) \geq \sigma(\widetilde{p}, \bar{g})$.

The assertion $\sigma(\widetilde{p}', \bar{g}) \geq \sigma(\widetilde{p}, \bar{g})$ implies that there exists a $\check{p} \in \mathcal{T}$ satisfying

$$\check{p} \leq \widetilde{p}', \quad |\check{p} - \widetilde{p}'| \geq \sigma, \quad g(\check{p}) \leq \bar{g}. \tag{61}$$

From $\widetilde{p} - \widetilde{p}' \geq 0$ and $\widetilde{p}' - \check{p} \geq 0$ we obtain:

$$\widetilde{p} \geq \check{p} \tag{62}$$

and:

$$|\check{p} - \widetilde{p}| = \sum_{i=1}^{N} |\widetilde{p}_i - \check{p}_i| \tag{63}$$

$$= \sum_{i=1}^{N} (\widetilde{p}_i - \check{p}_i) \qquad \text{(because } \widetilde{p} \geq \check{p}) \tag{64}$$

$$= \sum_{i=1}^{N}(\widetilde{p}_i - \widetilde{p}'_i) + \sum_{i=1}^{N}(\widetilde{p}'_i - \check{p}_i) \tag{65}$$

$$= \sum_{i=1}^{N}|\widetilde{p}_i - \widetilde{p}'_i| + \sum_{i=1}^{N}|\widetilde{p}'_i - \check{p}_i| \qquad \text{(because } \widetilde{p} \geq \widetilde{p}',\ \widetilde{p}' \geq \check{p}\text{)} \tag{66}$$

$$= |\widetilde{p} - \widetilde{p}'| + |\widetilde{p}' - \check{p}|. \tag{67}$$

Furthermore, $\widetilde{p}' \leq \widetilde{p}$ and $\widetilde{p}' \neq \widetilde{p}$ imply

$$|\widetilde{p}' - \widetilde{p}| > 0, \tag{68}$$

while in (61) we have that

$$|\widetilde{p}' - \check{p}| \geq \sigma. \tag{69}$$

Substitution of (68) and (69) into (67) yields

$$|\check{p} - \widetilde{p}| > \sigma, \tag{70}$$

Combining this with eq.(62), we can revise eq.(61) to:

$$\check{p} \leq \widetilde{p}, \quad |\check{p} - \widetilde{p}| > \sigma, \quad g(\check{p}) \leq \bar{g}, \tag{71}$$

Observe that (71) implies:

$$\exists \check{\sigma} > \sigma : \quad \check{p} \in \mathcal{D}(\check{\sigma}, \widetilde{p}), \quad g(\check{p}) \leq \bar{g}. \tag{72}$$

It follows from the definition of the weak redundancy, eq.(32), and eq.(72) that:

$$\sigma_1(\widetilde{p}, \bar{g}) \geq \check{\sigma} > \sigma,$$

which concludes the proof of the eq.(60), and this concludes the proof that eq.(42) implies eq.(43). ∎

10 References

1. Au, S.K. (2003). Discussion of "Redundancy Index of Lifeline Systems" by Masaru Hoshiya and Kinya Yamamoto, *Journal of Engineering Mechanics (ASCE)*, 129, 1104.

2. Ben-Haim Y. (2006). *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press, London.

3. Bertero R.D., Bertero V.V. (1999). Redundancy in earthquake-resistant design, *Journal of Structural Engineering (ASCE),* 125, 81–88.

4. Feng, Y.S., Moses, F. (1986). Optimum design, redundancy and reliability of structural systems. *Computers & Structures*, 24, 239–251.

5. Frangopol D.M., Curley J.P. (1987). Effects of damage and redundancy on structural reliability, *Journal of Structural Engineering (ASCE),* 113, 1533–1549.

6. Fu, G., Frangopol, D.M. (1990). Balancing weight, system reliability and redundancy in a multiobjective optimization framework. *Structural Safety*, 7, 165–175.

7. Hendawi, S., Frangopol, D.M. (1994). System reliability and redundancy in structural design and evaluation. *Structural Safety*, 16, 47–71.

8. Hoshiya, M. Yamamoto, K. (2002). Redundancy index of lifeline systems. *Journal of Engineering Mechanics (ASCE)*, 128, 961–968.

9. Hoshiya, M. Yamamoto, K. (2003). Closure to "Redundancy Index of Lifeline Systems" by Masaru Hoshiya and Kinya Yamamoto. *Journal of Engineering Mechanics (ASCE)*, 129, 1105.

10. Husain, M., Tsopelas, P. (2004). Measures of structural redundancy in reinforced concrete buildings. I: redundancy indices. *Journal of Structural Engineering (ASCE)*, 130, 1651–1658.

11. Liao, K.-W., Wen, Y.-K., Foutch, D.A. (2007). Evaluation of 3D steel moment frames under earthquake excitations. II: reliability and redundancy. *Journal of Structural Engineering (ASCE)*, 133, 471–480.

12. Ohi, K., Ito, T., Li, Z.-L (2004). Sensitivity on load carrying capacity of frames to member disappearance. *Proceedings of Council on Tall Building and Urban Habitat (CTBUH) Seoul Conference—Tall Buildings in Historical Cities: Culture & Technology for Sustainable Cities*, Seoul.

13. Okasha, N.M., Frangopol, D.M. (2009). Lifetime-oriented multi-objective optimization of structural maintenance considering system reliability, redundancy and life-cycle cost using GA. *Structural Safety*, 31, 460–474.

14. Paliou, C., Shinozuka, M., Chen, Y.-N. (1990). Reliability and redundancy of offshore structures. *Journal of Engineering Mechanics (ASCE)*, 116, 359–378.

15. Pandey, P.C., Barai, S.V. (1997). Structural sensitivity as a measure of redundancy. *Journal of Structural Engineering (ASCE)*, 123, 360–364.

16. Schafer, B.W., Bajpai, P. (2005). Stability degradation and redundancy in damaged structures. *Engineering Structures*, 27, 1642–1651.

17. Tesfamariam, S., Saatcioglu, M. (2010). Seismic vulnerability assessment of reinforced concrete buildings using hierarchical fuzzy rule base modeling. *Earthquake Spectra*, 26, 235–256.

18. Wen, Y.K., Song, S.-H. (2003). Structural reliability/redundancy under earthquakes. *Journal of Structural Engineering (ASCE)*, 129, 56–67.

19. Žiha, K. (2000). Redundancy and robustness of systems of events. *Probabilistic Engineering Mechanics*, 15(4), 347–357