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CONVEX MODELS OF UNCERTAINTY: APPLICATIONS AND IMPLICATIONS

ABSTRACT. Modern engineering has included the basic sciences and their accompanying mathematical theories among its primary tools. The theory of probability is one of the more recent entries into standard engineering practice in various technological disciplines. Probability and statistics serve useful functions in the solution of many engineering problems. However, not all technological manifestations of uncertainty are amenable to probabilistic representation. In this paper we identify the conceptual limitations of probabilistic and related theories as they occur in a wide range of engineering tasks. We discuss the structure and properties of an alternative, non-probabilistic, method – convex modelling – for quantitatively representing uncertain phenomena.

1. INTRODUCTION

Engineering sciences derive much of their inspiration and fundamental knowledge from the natural sciences. In particular, the engineers acquired the theory of probability fairly recently from the scientists (who got it from aristocratic 17th century gamblers!) We will illustrate the unsuitability of the theory of probability for some engineering applications, and will discuss the structure and properties of convex models of uncertainty.

Recent decades have witnessed a diversification of theories of uncertainty. These theories concentrate on quantification of uncertain information through the use of real-valued non-negative normalized mathematical functions. In place of the classical probability density one can today choose between functions of belief, possibility, necessity, provability and so on. In contrast, convex modelling emphasizes the structure of event sets, rather than the structure of measure functions defined on the space of events.

We begin by discussing the practical limitations of probability for a range of engineering tasks involving uncertainty (Section 2). We then briefly examine several engineering problems whose formulation and solution involve non-probabilistic models of uncertainty (Section 3). In Section 4 we describe the details of several convex models and their selection on the basis of engineering information. Then, in Section 5, we discuss a limit theorem which motivates the use of convex models,

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and compare this with the central limit theorem. Finally, in Section 6, we discuss a specific application of a convex model in some detail.

2. PRACTICAL LIMITATIONS OF PROBABILITY

Workers in various technological fields have noted that the practical limitation of probabilistic models for some purposes arises from the inability to verify the details of these models.

Consider the comment by Sobczyk and Spencer in discussing stochastic models of fatigue failure in structures. Referring to turbulent wind fluctuations acting on transport aircraft or tall buildings, Sobczyk and Spencer enumerate numerous complicating factors and conclude that "the engineering analysis of fatigue reliability assumes some *standard*[1] representations of the spectrum of a turbulent wind". Considering steel offshore platforms they assert that "the establishment of standard load spectra ... [is] much more difficult than for aircraft structures" ([38], p. 89). While the probabilistic models developed by Sobczyk and Spencer are empirically very useful, they are founded on more or less arbitrary stochastic assumptions.

In a similar vein, Murota and Ikeda develop a theory for buckling of trusses with geometrical imperfections, and comment that they

> have employed random imperfections ... although it is somewhat hypothetical at this stage, since the probability distribution cannot be known precisely in practice. The present analysis is not independent of the hypothetical distribution, and the quantitative aspects of the results will have limitations in applicability. However, the qualitative aspects of the conclusions will remain valid for a wide range of probability distributions [30].

Reliable engineering design would seem to depend on quantitative results, not only qualitative ones.

Concerning chaotic dynamical systems, Ekeland writes: "Randomness appears because the available information, though accurate, is incomplete. Part of the information is withheld from us" ([23] pp. 62–63). He comments that "... most deterministic systems are impossible to predict because their dynamics are too complicated for any meaningful computation to be possible" ([23], p. 107). The verification of the stochastic representation of complex dynamical systems is no less difficult.

3. CONVEX MODELLING: TECHNOLOGICAL EXAMPLES

Experiences such as these have led a number of engineering scientists to pursue non-probabilistic methods for representing uncertainty. In this section we discuss several technological situations which have given rise to the need for non-probabilistic representation of uncertainty. Our purpose is to delineate the context of problems within which the rest of the discussion emerges. We will discuss one particular non-probabilistic methodology: convex modelling.

Let us briefly outline the range of applications of non-probabilistic treatments of uncertainty, before proceeding to several more detailed cases. Drenick [16], [17] and Shinozuka [37] describe uncertain seismic loads on civil structures by defining sets of possible input functions. Schweppe [34] and Witsenhausen [45], [46] describe estimation and control algorithms for linear dynamic systems based on sets of unknown inputs. Schweppe [35] develops inference and decision rules based on assuming the uncertain phenomenon can be quantified in such a way as to be bounded by an ellipsoid. Ben-Haim and Elias [11] use convex models to represent uncertain heat flux variations. Ben-Haim [5] develops a method for optimal design of material assay systems based on convex sets of uncertain spatial distributions of the analyte material. Ben-Haim and Elishakoff [13] describe a range of analysis and design problems in applied mechanics based on defining convex sets of uncertain input functions or uncertain geometrical imperfections. Natke and Soong [31] study the topological optimization of mechanical structures with convex-model representation of uncertain dynamic loads on the structure. Lindberg [28], [29] and Ben-Haim [8] use the convex modelling method to study radial pulse buckling of thin-walled shells. Elishakoff and Cai [22] study the buckling of a column on a nonlinear elastic foundation, subject to uncertain initial imperfections which are represented by a convex model. Elishakoff and Colombi [21] use convex models to analyze the vibration of an acoustically excited structure with uncertain physical parameters. Givoli and Elishakoff [25] employ a convex model to examine stress concentrations in nearly circular holes with uncertain irregularities.

We now consider five examples of engineering analysis and design with uncertainty, based on convex models. A convex model is a convex set of functions.[2] Each function represents a possible realization of an uncertain event. In the examples that follow, these events are spatial distributions of imperfections in a building, or temporal variation of the ground motion during an earthquake, and so on.

Buckling of Thin-Walled Shells. Thin-walled shells such as cylinders and domes have very high load-bearing capacities compared to their weight. For example, a sheet of paper stood on end buckles under slight pressure, while if rolled into a cylinder and taped it can withstand an axial loading of considerable weight. (You can try it with several hefty books). However, small geometrical imperfections in the shape of the shell can drastically reduce the maximum load which the shell can carry.

A typical engineering question which arises is: what radial tolerance in the shell-shape assures that the weakest shell will suffer a reduction in buckling-load[3] by no more than a specified amount? In a more sophisticated analysis one recognizes that the boundary conditions of the shell, together with the shell dynamics, can allow greater tolerance in some regions of the shell than others. Consequently, one can ask: what variation of the shell-shape tolerance over the surface of the shell is allowed?

The first problem one confronts in addressing these questions is: how to model the range of possible shell shapes coming off a production line? Some information is available from actual measurements of shell shapes, though it is scanty and very expensive (see, for example, [1], [27]). However, one can readily formulate an infinite set of functions which represents the uncertainty of the shapes: each function represents a particular shell shape, while the set expresses the uncertainty in which shape will actually occur. This is precisely a convex model for shell-shape uncertainty. Furthermore, one can do so in such a way that the set depends parametrically on the radial tolerance of the shell. Then it is possible to evaluate the buckling load of the weakest shell as a function of the radial tolerance [12], [13], [20]. In this way, the convex model incorporates uncertainty in the shell shape into the design and manufacturing procedure, without relying on probabilistic information.

Vehicle Vibrations on Rough Terrain. A vehicle traversing rough terrain can induce discomfort or even functional incapacity in the passengers as well as damage to on-board equipment. The design of the suspension system must be optimized to reduce these effects. The optimization must be performed with respect to a model of the uncertain terrain. Statistical models have been employed for representing the variation of uncertain terrain [32]. However, verification of these models is time consuming and expensive. Alternatively, global features of the surface which are comparatively easily measured, such as maximum roughness or slope variation or other features, can be used to define sets of possible substrates. These sets are convex models of the substrate uncertainty. The design decisions are then made so as to assure that the worst ride (e.g. maximum instantaneous acceleration) induced by any allowed substrate is acceptable [13], [14].

Seismic Safety. A major challenge in civil engineering design for seismically active regions is the prevention of life-threatening structural damage resulting from earthquakes. Also important is the amelioration of seismically-induced damage to equipment and secondary systems (power systems, communications equipment, etc.). The design of seismically safe structures is complicated by the wide temporal and spatial variability of ground motion during an earthquake and its complex interaction with structures.

Probabilistic models have been used in recent decades to represent the uncertainty of vibrating structures [19], [39] and seismic ground motion [40], [47]. The concern about these models arises from the fact that a stochastic model represents typical events much more reliably than rare events, especially when the model is based on limited information. In discussing fatigue failure of offshore structures, Hartt and Lin comment that it is the "extreme, infrequent stress excursions which may be important either with regard to direct damage or to subsequent interaction effects" ([26], p. 91). Rare events in probabilistic models are described by the tails of the distribution, while probability distributions are usually specified in terms of mean and mean-variation parameters. This makes probabilistic models risky design tools, since it is rare events, the catastrophic ones, which must underlie the reliable design.

Global features of seismic events, such as total or instantaneous energy, can be used to define sets of seismic events which include extreme cases more explicitly than probabilistic models. It is fairly straightforward to formulate a convex model as the set of all seismic events consistent with available fragmentary data constraining the seismic occurrence. One can then optimize the design with respect to this set of conceivable events. In this way information about earthquake-uncertainty is incorporated in the structural design-decisions without postulating stochastic properties of seismic phenomena [16], [17], [37].

Assay of Pulmonary Aerosols. A challenging and important healthmonitoring measurement in the nuclear industry is the *in vivo* assay of plutonium aerosols in human lungs. The assay is performed by passively detecting the L X-rays of plutonium; the quantity of plutonium is proportional to the intensity of the detected radiation. A major source of uncertainty is the spatial distribution of the plutonium in the pulmonary tissue. The measurement is very sensitive to this spatial distribution because the distance over which the radiation-intensity is reduced by half is on the order of 1 cm: much less than the dimensions of the lung and thoracic cavity. In other words, different spatial distributions of the same quantity of analyte can result is measured radiation-intensities which differ by orders of magnitude.

Numerous factors, ontogenetic, physiological and environmental, determine the spatial distribution of the analyte. The task of verifying a meaningful probabilistic model of the spatial distribution is enormous. Nevertheless, the assay system must be constructed to assure detection and quantitative measurement of minute quantities of this pernicious poison. This can be achieved by quantifying the spatial uncertainty with a convex model: a set of allowed spatial distributions of the analyte [5]. Furthermore, adaptive assay algorithms can also be implemented which augment the reliability of the measurement [43].

Design of a Reliable Pressure Vessel. A standard task in structural engineering design is to choose the wall thickness of a pressure vessel subject to uncertain internal fluid pressure. If the probability density function (pdf) of the pressure fluctuations is known, then it is a matter of fairly straightforward analysis to determine the least wall thickness needed to assure that the probability of failure by yielding is less than a specified amount.[4] However, if high reliability is required (low probability for failure), then even very small errors in the tails of the probability density can result in large errors in the chosen wall thickness ([10]; [13], pp. 11–13). A hybrid probabilistic–non-probabilistic approach is possible here. One defines the set of all pdfs which are consistent with available information. This set is a convex model for the uncertainty in the pdf. The wall thickness is then chosen with respect to this non-probabilistic specification of the uncertainty in the pdf of the pressure.

4. SOME CONVEX MODELS

In this section we define a range of convex models of uncertainty, and discuss typical engineering considerations underlying the selection of a model.

Energy-Bound Models. Consider a warped beam which deviates by $y(x)$ from its nominal shape at position x along the length of the beam. Energy is required to straighten out such a beam. Specifying the amount of energy required still leaves some uncertainty as to the original shape of the beam. One type of energy-bound convex model of the shape-uncertainty is defined as the set of all beam-profiles requiring no more than $\mathscr{E}$ of elastic energy to straighten them out [9]. This set of profiles is:

$$Y(\mathscr{E}) = \left\{ y(x) \colon \frac{EI}{2} \int_0^L (\ddot{y}(x))^2 \, dx \leqslant \mathscr{E} \right\} \tag{1}$$

where L and EI are the length and flexural rigidity of the beam, respectively, and dots imply differentiation with respect to position. (It is implicitly assumed that the elements of $Y(\mathscr{E})$ satisfy the boundary conditions inherent in the mechanical system.)

Energy-bounds can be related to uncertainty in many situations. In Section 3 we mentioned energy-bound models representing uncertainty of seismic input waveforms, in terms of bounds on the total or instantaneous ground-motion energy. Not infrequently the 'energy' is loosely defined and, in analogy to the energy of an electric current, the convex model is defined as a bound on a quadratic function. For example, if $u(t)$ is a scalar function representing the uncertain ground motion as a function of time, a common energy-bound uncertainty model is:

$$U(\mathscr{E}) = \left\{ u(t) \colon \int_0^T u^2(t) \, dt \leqslant \mathscr{E} \right\}. \tag{2}$$

An enormously popular 'energy-bound' uncertainty model is the ellipsoidal bound model, studied extensively by Schweppe [35]. If v is the uncertain vector, then an ellipsoidal model for the uncertainty in v is the set of vectors contained within an ellipsoid:

$$V(\mathscr{E}) = \{ v \colon v^T W v \leqslant \mathscr{E} \} \tag{3}$$

where W is a real symmetric positive definite matrix. For example, an ellipsoidal model of uncertainty can be used to represent uncertain geometrical imperfections in the shape of a shell [20].

Envelope-Bound Models. The need to represent geometric uncertainties, as well as other applications, gives rise to envelope-bound convex models. Forces acting on an unknown domain of a structure [6], or obstacles of unknown size and position in air ducts [7], or local imperfections in shells [8] are all amenable to representation by envelope-bound models. Let $g(x)$ be the uncertain function of a spatial variable x. An envelope-bound model is:

$$E(g_1, g_2) = \{g(x): g_1(x) \leqslant g(x) \leqslant g_2(x)\} \tag{4}$$

where $g_1(x)$ and $g_2(x)$ define the bounding envelope. To take a specific example, consider a beam of length L, so $0 \leqslant x \leqslant L$. Suppose the beam is warped in the interval $[a, b]$ but otherwise straight, and let $g(x)$ represent the imperfection profile of the beam. By choosing the envelope functions g_1 and g_2 as follows, $E(g_1, g_2)$ can represent uncertain local damage of magnitude not exceeding δ:

$$g_n(x) = \begin{cases} 0 & x \notin [a, b] \\ (-1)^n \delta & x \in [a, b] \end{cases} \quad n = 1, 2. \tag{5}$$

Slope-Bound Models. The envelope-bound concept can be applied to the slope rather than to the magnitude of a spatially uncertain quantity. Such convex models have been used in analysis of vehicle dynamics on barriers and uncertain rolling terrain [14]. Similarly, in modelling uncertain heating processes [11] the uncertain function may be constrained to increase monotonically between given limits, but to be otherwise of unknown variation.

For example, let $r(t)$ be the heat flux out of a nuclear reactor fuel element which, during a transient, increases monotonically between r_1 and r_2 over the time interval $[0, T]$. A slope-bound convex model for the uncertainty in $r(t)$ during the transient is:

$$R(s) = \left\{ r(t): r(0) = r_1, r(T) = r_2, \frac{\mathrm{d}r}{\mathrm{d}t} \geqslant 0 \right\}. \tag{6}$$

Fourier-Bound Models. In many situations the engineer has partial spectral information for characterizing an uncertain phenomenon. For example, geometric shape-imperfections of thin walled shells, mentioned in Section 3, have been measured spectrally [1], [27]. Data such as these lead to ellipsoidal-bound models for the uncertainty in the spectral coefficients. Let c represent a vector of Fourier coefficients of the shape of the geometric imperfection. A Fourier ellipsoidal-bound convex model of uncertainty is [8], [28], [29]:

$$(7) \qquad C(\rho) = \{c: (c - \bar{c})^T W (c - \bar{c}) \leqslant \rho^2\}$$

where $\bar{c}$ is a nominal Fourier-coefficient vector and W is a matrix determining the shape of the ellipsoid.

Spectral envelope-bound models are also used. If $u(\omega)$ is an uncertain Fourier transform, then a Fourier envelope-bound model is [10]:

$$(8) \qquad U(u_1, u_2) = \{u(\omega): u_1(\omega) \leqslant |u(\omega)| \leqslant u_2(\omega)\}$$

where $|u(\omega)|$ is the absolute value of the complex function $u(\omega)$, and $u_1(\omega)$ and $u_2(\omega)$ are real envelope functions.

Mass Distribution Models. In the assay of material it sometimes occurs that very little is known about the possible spatial distributions which the material can assume. Such problems arise in nuclear radiological measurements [43] as mentioned in Section 3, in nuclear waste assay [5], [15], [36], in subterranean geological prospecting [44] and elsewhere. The simplest convex model for representing unknown spatial distributions of material is the distribution-function model. Let $m(x)$ be the density of analyte material at position x, distributed over domain X. The set of allowed distributions of total mass μ_0 is:

$$(9) \qquad M_0(\mu_0) = \left\{ m(x): m(x) \geqslant 0, \quad \int_X m(x)\, \mathrm{d}x = \mu_0 \right\}.$$

In some situations, the information which constrains the allowed spatial distributions of analyte material is the nth (usually 1st or 2nd) moments of the spatial distribution. In this case, a convex model for uncertainty in $m(x)$ is:

$$(10) \qquad M_n(\mu_n) = \left\{ m(x): m(x) \geqslant 0, \quad \int_X x^n m(x)\, \mathrm{d}x = \mu_n \right\}.$$

5. CONVEX MODELS: A MOTIVATION

The convex models of uncertainty surveyed in the previous section are all *convex sets* of functions. From the engineering point of view, each of these sets is defined as the collection of all elements consistent with a given quantity of information (an energy bound, a spectral envelope, and so on). The convexity of these sets arises 'naturally', as a by-product.

It is clear that set-models of uncertainty are not by necessity convex; important situations arise in which the sets involved are not convex. However, the following elementary theorem [13] provides some indication of why convexity is not just accidental in the modelling of uncertainty.

Let $f(t)$ be a time (or space) varying uncertain vector function, and let Γ be a set of such functions. For a positive integer n, consider the set of functions constructed as n-fold averages of elements of Γ:

$$(11) \qquad F_n = \left\{ f \colon f(t) = \frac{1}{n} \sum_{m=1}^{n} g_m(t), \quad \text{for all } g_m \in \Gamma \right\}.$$

It is well known that, as $n \to \infty$, the sequence of sets F_n, $n = 1, 2, \ldots$ converges to the convex hull[5] of Γ:

$$(12) \qquad \lim_{n \to \infty} F_n = \mathrm{ch}(\Gamma).$$

(For more general results see [2], [3], [4]).

This theorem suggests the following physical interpretation. If an uncertain, macroscopic process, represented by $f(t)$, is formed as the linear superposition of numerous microscopic processes $g_m(t)$, each drawn from the set Γ, then the set of all processes $f(t)$ will tend to be convex, regardless of the structure of the set Γ. In other words, we might expect that complex vector processes will tend to cluster in convex sets of functions.

This result bears a suggestive similarity to the central limit theorem, even though the contents and proofs of these two theorems differ utterly. Let $g_1, g_2, \ldots$ be independent, identically distributed random variables. Technicalities aside, the central limit theorem states that, as $n \to \infty$, the distribution of the sum

$$f = \frac{1}{\sqrt{n}} \sum_{m=1}^{n} g_m$$

converges to a normal distribution, regardless of how the g_m's are distributed. The physical implication is that if a macroscopic random quantity f is composed of a multitude of superimposed independent random microscopic quantities g_m, then f should tend to display a normal distribution.

The central limit theorem and Equation (12) both relate fairly arbitrary microscopic uncertainties to rather more specific macroscopic uncertainty models. Despite this similarity, however, the points of emphasis of these two results are completely disparate. The central limit theorem directs attention to the structure of the *probability measure*, while Equation (12) focusses on the structure of the *event set*.

Historically speaking, the proof of the central limit theorem, presented in 1810 by Laplace, was a great advance in understanding the fundamental mathematical nature of probability densities [41]. In addition, the theorem provided a justification of the least-squares estimation method developed five years before by Legendre and, independently, by Gauss. Moreover, the central limit theorem directed the attention of researchers to probability densities and their derivations. This led, in the course of the 19th century, to the discovery of other statistical distributions.

In recent decades attention has been placed on extending the concept of probability density. In the place of classical probability functions one has membership functions and measures of possibility and necessity in fuzzy logic [18], belief functions in the Dempster-Shafer theory [33], and so on. The logical diversity of these theories is real and substantial, as evidenced by the distinct axiomatic bases on which they rest [24]. However, the intellectual connection to traditional uncertainty models is clear: modern as well as classical thought concentrates on the properties and structure of normalized non-negative functions defined on sets of events.

In contrast, set-theoretical models of uncertainty, such as convex models, concentrate on the geometric structure of event-clusters. What has attracted the attention of workers in various technological areas is the fact that fragmentary information about uncertain events often leads to the definition of a convex set of events. This provides both a standardized framework for analysis, as well as a guide to the formulation of plausible uncertainty models based upon severe lack of prior information.

6. DETAILED EXAMPLE: RELIABILITY OF AXIALLY-LOADED SHELLS WITH INITIAL GEOMETRICAL IMPERFECTIONS

In this section we will use a convex model to perform a reliability-analysis with respect to structural uncertainty. We consider an axially-compressed thin-walled shell with initial geometrical imperfections.

The shell length is L. The axial coordinate, along the length of the shell, is z, which we normalize as $\xi = \pi z/L \in [0, \pi]$. The azimuthal coordinate is $\theta \in [0, 2\pi]$. The deviation of an actual shell from the nominal shell dimension at point (ξ, θ) is $\eta(\xi, \theta)$. We represent the set of allowed imperfection-functions by the uniform-bound convex model:

$$(13) \qquad F(\hat{\eta}) = \{\eta(\xi, \theta): |\eta(\xi, \theta)| \leq \hat{\eta}\}.$$

The deviations from the nominal shell shape are uniformly bounded by $\hat{\eta}$: every imperfection-function, $\eta(\xi, \theta)$, whose magnitude nowhere exceeds $\hat{\eta}$, is included in F. One can view $\hat{\eta}$ as a radial tolerance of the shells whose imperfections are represented by F.

A typical question which arises in design-for-reliability is: how large a radial tolerance is acceptable, when the shell will bear static axial loads up to the value $\lambda_{\max}$?

Implicit in this question is a statement about the uncertainty in the actual shell shapes. If in fact the designer knows nothing about the geometrical imperfections other than the value of the radial tolerance to which the shells will be produced, then F is probably the most detailed representation of the range of possible shell shapes which can be justified by the available data. If additional information is available, such as spectral data about the spatial frequencies of the imperfections, then other convex models would be appropriate. Various more detailed convex models for this purpose are discussed in [8], [12], [13], [28], [29].

We will proceed with the simple uniform-bound convex model. The design question can be formulated as follows. The design-parameter is $\hat{\eta}$, the radial tolerance. For any shell, the lowest applied load which causes the shell to fail by buckling is the 'buckling load'. Denote by $\mu(\hat{\eta})$ the least buckling load of any shell in $F(\hat{\eta})$. Then determine the greatest value of the radial tolerance, $\hat{\eta}$, for which the least buckling load, $\mu(\hat{\eta})$, exceeds the maximum applied load, $\lambda_{\max}$.

The mechanical analysis of geometrically imperfect shells is most conveniently done when the imperfections are expressed in terms of their Fourier coefficients. Let $x(\eta)$ be a vector of the dominant spatial Fourier coefficients of $\eta(\xi, \theta)$. Let x^0 be the vector of Fourier coefficients of the nominal shell shape. Let $\Psi(x^0)$ be the buckling load of this nominal shell, and $\Psi(x^0 + x)$ be the buckling load of a shell with initial imperfections whose Fourier coefficients are x. For small imperfections we can expand $\Psi(x^0 + x)$ as:

$$\Psi[x^0 + x(\eta)] = \Psi(x^0) + x^T(\eta) \left.\frac{\partial \Psi}{\partial x}\right|_{x=x^0}. \tag{14}$$

Some manipulations lead to the following expression for the reduced buckling load due to the imperfection function $\eta(\xi, \theta)$:

$$\Psi[x^0 + x(\eta)] = \Psi(x^0) + \int_0^{2\pi} \int_0^{\pi} \eta(\xi, \theta) S(\xi, \theta)\, \mathrm{d}\xi\, \mathrm{d}\theta \tag{15}$$

where $S(\xi, \theta)$ is a combination of trigonometric functions with coefficients which depend on the elements of the vector $\partial \Psi(x = x^0)/\partial x$. See [12], [13].

Examination of Equation (15) shows that the greatest reduction in the buckling load is obtained from the imperfection-function which switches between its extreme values, $+\hat{\eta}$ and $-\hat{\eta}$, as $S(\xi, \theta)$ changes sign from negative to positive. The minimum buckling, load for shells whose radial tolerance is $\hat{\eta}$, is:

$$\mu(\hat{\eta}) = \min_{\eta \in F} \Psi[x^0 + x(\eta)] \tag{16}$$

$$= \Psi(x^0) - \hat{\eta} \int_0^{2\pi} \int_0^{\pi} |S(\xi, \theta)|\, \mathrm{d}\xi\, \mathrm{d}\theta. \tag{17}$$

This relation expresses the buckling load of the weakest shell from among the ensemble of shells whose radial tolerance is $\hat{\eta}$. Additionally, it is based on mechanical data expressing the imperfection-sensitivity of the buckling load, which appears in the function $S(\xi, \theta)$. Equation (17) is derived for small imperfections, and is therefore linear in the parameter $\hat{\eta}$.

Equation (17) underlies the convex-modelling assessment of the reliability of the uncertain shell. The shell uncertainty is expressed by $\hat{\eta}$ and the range of performance – embodied in the least buckling load –

is given by $\mu(\hat{\eta})$. One chooses the radial tolerance to assure that the maximum axial load does not exceed the least buckling load:

$$\lambda_{\max} < \mu(\hat{\eta}). \tag{18}$$

Uncertainty plays a central role in this analysis: F represents a set of shells, any one of which could occur. Any given physical shell with tolerance not exceeding $\hat{\eta}$ is represented by one of the imperfection functions in F; which one, one does not know. The shell is 'reliable' in the sense of our non-probabilistic model of uncertainty when (18) is satisfied.

While uncertainty in the shell shapes is fundamental to this analysis, there is no likelihood information, either in the formulation of the convex model or in the concept of reliability. It might be useful, for example, to summarize the reliability of a given radial tolerance by a quantity such as:[6]

$$r = 1 - \frac{\lambda_{\max}}{\mu(\hat{\eta})}. \tag{19}$$

When r is close to unity, the maximum applied load ($\lambda_{\max}$) is far less than the least load-bearing capacity ($\mu(\hat{\eta})$); the system is 'reliable' in the non-probabilistic sense. As r approaches zero, the maximum applied load approaches the least buckling load, and failure becomes more imminent. However, unlike in a probabilistic analysis, r has no connotation of likelihood. We have no rigorous basis for evaluating how likely failure may be; we simply lack the information, and to make a judgement would be deceptive and could be dangerous. There may definitely be a likelihood of failure associated with any given radial tolerance. However, the available information does not allow one to assess this likelihood with any reasonable accuracy.

7. CONCLUSION

The quantitative representation of complex uncertain phenomena engages the attention of workers in many technological fields. Severe lack of information makes the verification of probabilistic models difficult or impractical in some situations. This has led to the development of quantitative uncertainty models in which the prior information about the uncertainty is invested in formulating the structure of event sets.

This emphasis on the structure of event sets is in contrast to traditional probability theory as well as the developments of recent decades in uncertainty modelling, like fuzzy logic or Dempster-Shafer theory, which emphasize a normalized real function defined on the space of events.

It has been found that, with the type of information often available, these set-models of uncertainty often lead to convex sets. In this way, many technological problems are formulated, analyzed and solved with full cognition of the uncertainties involved but without invoking probabilistic thinking. Instead, set-theoretical convex models represent uncertainty in terms of indeterminacy. (Suppes and Zanotti discuss a related motivation for indeterminacy [42].)

In formulating a convex model for design-for-reliability, the engineer starts with the realization that the conditions under which his artefact will operate are incompletely characterized. He then asks for the set of all conditions consistent with what *is* known, and then chooses his design so that acceptable performance is anticipated within this set of "possible worlds". His main task in formulating the uncertainty model is to choose a set-structure which adequately reflects the constraints defining the uncertainties.

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NOTES

[1] Italics occur in the original.

[2] A set S is convex if, for all elements f and g in S and all numbers $0 < \alpha < 1$, the quantity $\alpha f + (1 - \alpha)g$ also belongs to S.

[3] The buckling load is the least load at which the shell fails by buckling.

[4] Things can get complicated if one insists on including such inconveniences as chemical corrosion, stress corrosion cracking, material imperfections, or other unpredictable embarrassments.

[5] The convex hull of a set S is the intersection of all convex sets containing S. Consequently, ch(S) is a convex set. Roughly speaking, ch(S) is the "smallest" convex set containing S.

[6] I am indebted to Prof. C. Cempel, Technical University of Poznan, Poland, for this suggestion.

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