

Lecture 2

Info-Gap Robustness of a Beam

with

Uncertain Load

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1 Info-Gap Robustness of a Beam With an Uncertain Load

(Source: Yakov Ben-Haim, 1996, *Robust Reliability in the Mechanical Sciences*, Springer, sections 3.1, 3.2.)

¶ 3 components of reliability analysis:

1. A system model.
2. A failure criterion.
3. An uncertainty model.

¶ We will consider info-gap models of uncertainty and develop, in a preliminary example, the idea of **info-gap robustness**.

¶ Consider a:

- Uniform simply-supported beam.
- Uncertain distributed load density function, $\phi(x)$ [N/m].

¶ We wish to

- Analyze the reliability of the beam given very fragmentary information.
- Optimize the design of the beam by enhancing the reliability.
- Evaluate the impact of different levels and types of information.

¶ What we **do know** about the load:

- $\tilde{\phi}(x)$ = nominal load density function, [N/m].
- Substantial deviation from the nominal load is bounded along the beam.

¶ What we **do not know** about the load:

- The precise realization of the load density, $\phi(x)$.
- The bound on the deviation of the true from the nominal load.

¶ The disparity between what we

do know and what we **need to know**
for a fully competent design or analysis
is an **information gap**.

¶ We represent the load uncertainty with an info-gap model:

$$\mathcal{U}(h, \tilde{\phi}) = \left\{ \phi(x) : \left| \phi(x) - \tilde{\phi}(x) \right| \leq h \right\}, \quad h \geq 0 \quad (1)$$

This is an info-gap **uncertainty model**.

¶ Note the two levels of uncertainty in an info-gap model:

- At fixed h : true load profile $\phi(x)$ is unknown.
- Horizon of uncertainty — h — is unknown.

¶ **2 properties of all info-gap models:**

- *Contraction:*

$$\mathcal{U}(0) = \{\tilde{\phi}(x)\} \tag{2}$$

- *Nesting:*

$$h < h' \implies \mathcal{U}(h) \subseteq \mathcal{U}(h') \tag{3}$$

¶ **System model:**

- Static bending moment as a function of load profile: $M(x)$.
- For simple-simple beam one finds:

$$M(x) = -\frac{L-x}{L} \int_0^x \phi(u)u \, du - \frac{x}{L} \int_x^L \phi(u)(L-u) \, du \tag{4}$$

where L is the length of the beam.

¶ **The failure criterion:**

The beam fails if the bending moment $M(x)$ exceeds the critical value M_c :

$$\max_{0 \leq x \leq L} |M(x)| > M_c \tag{5}$$

¶ We evaluate the **robustness**, \hat{h} , by combining

System model, uncertainty model, and failure criterion:

The **robustness** is:

The greatest info-gap, h ,
such that the **system model**
does not violate the **failure criterion**
for any load profile up to **uncertainty** h .

We can express \hat{h} as:

$$\hat{h} = \text{maximum tolerable uncertainty} \tag{6}$$

$$= \max \{h : \text{failure cannot occur}\} \tag{7}$$

$$= \max \left\{ h : \left(\max_{0 \leq x \leq L} |M(x)| \right) \leq M_c \text{ for all } \phi(x) \text{ in } \mathcal{U}(h, \tilde{\phi}) \right\} \tag{8}$$

$$= \max \left\{ h : \left(\max_{\phi \in \mathcal{U}(h, \tilde{\phi})} \max_{0 \leq x \leq L} |M(x)| \right) \leq M_c \right\} \tag{9}$$

We can invert the order of the maxima inside the set.

¶ We begin by evaluating:

$$\max_{\phi \in \mathcal{U}(h, \tilde{\phi})} |M(x)| = \max \left(\max_{\phi \in \mathcal{U}(h, \tilde{\phi})} M(x), \left| \min_{\phi \in \mathcal{U}(h, \tilde{\phi})} M(x) \right| \right) \quad (10)$$

¶ To find these extrema note that:

- Other than $\phi(u)$, the integrands of both integrals in eq.(4) on p.4 have the same sign everywhere.
- Thus, extremal $M(x)$ is obtained by choosing $\phi(x) = \tilde{\phi}(x) + h$ or $\phi(x) = \tilde{\phi}(x) - h$.
- **We consider a special case:** $\tilde{\phi}(x) =$ positive constant.
- The results:

$$\max_{\phi \in \mathcal{U}(h, \tilde{\phi})} M(x) = -\frac{(h - \tilde{\phi})x(L - x)}{2} \quad (11)$$

$$\min_{\phi \in \mathcal{U}(h, \tilde{\phi})} M(x) = -\frac{(h + \tilde{\phi})x(L - x)}{2} \quad (12)$$

Hence:

$$\max_{\phi \in \mathcal{U}(h, \tilde{\phi})} |M(x)| = \frac{(h + \tilde{\phi})x(L - x)}{2} \quad (13)$$

¶ We are now ready to evaluate the second optimization, on x , in the expression for the robustness, eq.(9) on p.4.

We find the maximum at $x = L/2$, resulting in:

$$\max_{0 \leq x \leq L} \max_{\phi \in \mathcal{U}(h, \tilde{\phi})} |M(x)| = \frac{(h + \tilde{\phi})L^2}{8} \quad (14)$$

¶ The robustness is the greatest h at which the maximum bending moment $M(x)$ does not exceed the critical value M_c .

We find:

$$\underbrace{\frac{(h + \tilde{\phi})L^2}{8}}_{\text{max bending moment}} = \underbrace{M_c}_{\text{critical moment}} \implies \hat{h} = \frac{8M_c}{L^2} - \tilde{\phi} \quad (15)$$

Design implications: the robustness, \hat{h} , increases as:

- The beam length L decreases.
- The nominal load $\tilde{\phi}$ decreases.
- The critical bending moment M_c increases.

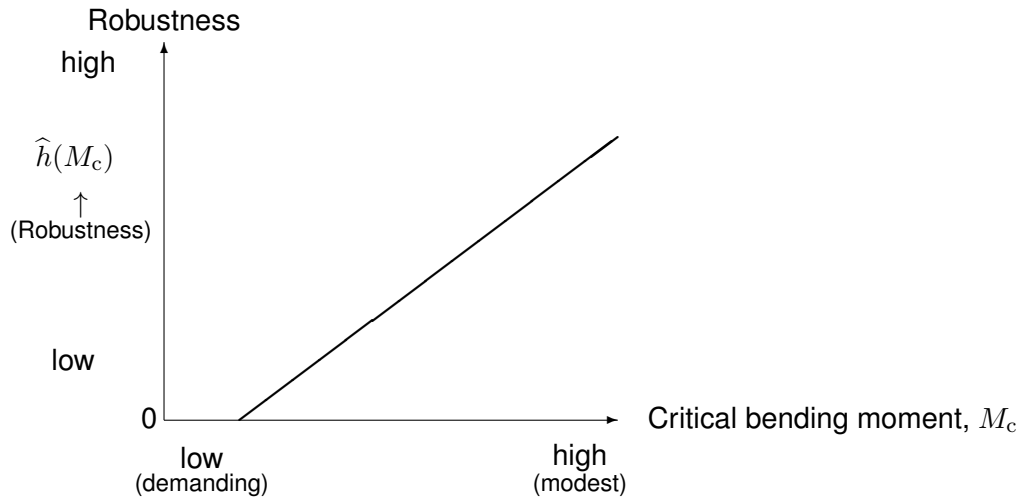


Figure 1: Robustness curve.

¶ **Two Properties:** Trade-off and zeroing (see fig. 1).

¶ **Trade off:** robustness vs performance.

- $\hat{h}(M_c)$ gets worse (decreases) as M_c gets better (decreases).
- This is sometimes called the pessimist's theorem. Why?
- The slope of the robustness curve expresses the cost of robustness. Why?

¶ **Zeroing:** Estimated performance has zero robustness:

$$\hat{h}(M_c) = 0 \quad \text{if} \quad M_c = \frac{\tilde{\phi}L^2}{8} = \text{estimated bending moment} \quad (16)$$

2 Statically Loaded Beam: Continued

¶ Knowledge is:

- Power.
- Robustness against surprise and uncertainty.

2.1 Load-Uncertainty Envelope

¶ **Different prior information; different uncertainty.** Examples:

- Hidden load on left half of beam.
- Flow perpendicular to beam; increasing turbulence in middle region.

¶ Let us now consider different prior information.

Rather than the uniform-bound info-gap model of eq.(1) on p.3, suppose we have information which indicates that the uncertain deviation $\phi(x) - \tilde{\phi}(x)$ varies within an envelope:

$$\mathcal{U}(h, \tilde{\phi}) = \left\{ \phi(x) : \left| \phi(x) - \tilde{\phi}(x) \right| \leq h\psi(x) \right\}, \quad h \geq 0 \quad (17)$$

where we **know**:

$\tilde{\phi}(x)$ = nominal load profile.

$\psi(x)$ = load-uncertainty envelope.

and we **do not know**:

$\phi(x)$ = actual load profile.

h = uncertainty parameter, horizon of uncertainty.

¶ **Examples of envelope function, $\psi(x)$:**

- Hidden load on left half of beam.

$$\psi(x) = \begin{cases} 1, & 0 \leq x \leq L/2 \\ 0, & L/2 < x \leq L \end{cases} \quad (18)$$

- Flow perpendicular to beam; increasing turbulence in middle region.

$$\psi(x) = \sin \frac{\pi x}{L} \quad (19)$$

¶ As usual with an info-gap model, there are two levels of uncertainty:

- Unknown realization $\phi(x)$ at info-gap h .
- Unknown horizon of uncertainty, h .

¶ As before:

- The system model is eq.(4) on p.4.
- The failure criterion is eq.(5) on p.4.

¶ To find the maximum absolute bending moment

we evaluate the max and the min of $M_\phi(x)$.

The max (least negative) is obtained with the lowest possible load profile, while

The min (most negative) is obtained with the greatest possible load profile.

We find:

$$M_1(x) = \min_{\phi \in \mathcal{U}(h, \tilde{\phi})} M(x) \quad (20)$$

$$= -\frac{L-x}{L} \int_0^x [\tilde{\phi}(u) + h\psi(u)] u \, du - \frac{x}{L} \int_x^L [\tilde{\phi}(u) + h\psi(u)] (L-u) \, du \quad (21)$$

$$M_2(x) = \max_{\phi \in \mathcal{U}(h, \tilde{\phi})} M(x) \quad (22)$$

$$= -\frac{L-x}{L} \int_0^x [\tilde{\phi}(u) - h\psi(u)] u \, du - \frac{x}{L} \int_x^L [\tilde{\phi}(u) - h\psi(u)] (L-u) \, du \quad (23)$$

We can express these succinctly as:

$$M_1(x) = M_{\tilde{\phi}}(x) + hM_\psi(x) \quad (24)$$

$$M_2(x) = M_{\tilde{\phi}}(x) - hM_\psi(x) \quad (25)$$

where $M_{\tilde{\phi}}(x)$ and $M_\psi(x)$ are defined implicitly in eqs.(21) and (23).

¶ Let us consider a **special case**:

The nominal load increases towards the center of the beam:

$$\tilde{\phi}(x) = \tilde{\phi} \sin \frac{\pi x}{L} \quad (26)$$

where $\tilde{\phi}$ is a known positive constant.

The uncertainty in the load increases towards the center of the beam:

$$\psi(x) = \sin \frac{\pi x}{L} \quad (27)$$

¶ Note that $\phi(x)$, $\tilde{\phi}(x)$ and h all have the same units.

The functions in eqs.(24) and (25) become:

$$M_{\tilde{\phi}}(x) = -\frac{L^2\tilde{\phi}}{\pi^2} \sin \frac{\pi x}{L} \quad (28)$$

$$M_{\psi}(x) = \frac{M_{\tilde{\phi}}(x)}{\tilde{\phi}} \quad (29)$$

¶ The least and greatest bending moments at point x are:

$$M_1(x) = -(\tilde{\phi} + h)\frac{L^2}{\pi^2} \sin \frac{\pi x}{L} \quad (30)$$

$$M_2(x) = -(\tilde{\phi} - h)\frac{L^2}{\pi^2} \sin \frac{\pi x}{L} \quad (31)$$

¶ From this we find that the greatest absolute bending moment occurs at the midpoint of the beam:

$$\max_{0 \leq x \leq L} \max_{\phi \in \mathcal{U}(h, \tilde{\phi})} |M(x)| = \frac{(\tilde{\phi} + h)L^2}{\pi^2} \quad (32)$$

¶ To find the robustness, we equate the maximum bending moment to the critical moment and solve for h :

$$\frac{(\tilde{\phi} + h)L^2}{\pi^2} = M_c \implies \hat{h} = \frac{\pi^2 M_c}{L^2} - \tilde{\phi} \quad (33)$$

This is quite similar to the uniform-bound case, eq.(15) on p.5.

¶ **The two info-gap models we have studied are:**

$$\mathcal{U}(h, \tilde{\phi}) = \left\{ \phi(x) : \left| \phi(x) - \tilde{\phi}(x) \right| \leq h \right\}, \quad h \geq 0 \quad (34)$$

(Eq.(1) on p. 3.) with robustness (eq.15), p.5:

$$\hat{h} = \frac{8M_c}{L^2} - \tilde{\phi} \quad (35)$$

$$\mathcal{U}(h, \tilde{\phi}) = \left\{ \phi(x) : \left| \phi(x) - \tilde{\phi}(x) \right| \leq h\psi(x) \right\}, \quad h \geq 0 \quad (36)$$

(Eq.(17) on p. 7) with robustness in eq.(33):

$$\hat{h} = \frac{\pi^2 M_c}{L^2} - \tilde{\phi} \quad (37)$$

• Both of these uncertainty models entail **unbounded rate of variation**.

• We sometimes have information which constrains the rate of variation of the uncertain function. We will now develop the tools needed to exploit this information.

2.2 Fourier Representation of a Function

¶ We interrupt our study of this example to briefly introduce the Fourier representation of a function. We will use Fourier representations in a new type of info-gap model.

¶ Motivation:

- The info-gap models of eqs.(1), p.3, and (17), p.7, allow unbounded rate of variation.
- We now have new information that constrains the rate of variation.

¶ Let $\phi(x)$ be an arbitrary but piece-wise continuous function defined on the interval $-L \leq x \leq L$. Then $\phi(x)$ can be represented as:

$$\phi(x) = \sum_{n=0}^{\infty} \left[b_n \sin \frac{n\pi x}{L} + c_n \cos \frac{n\pi x}{L} \right] \quad (38)$$

¶ Let $\phi(x)$ be an arbitrary but piece-wise continuous function defined on the interval $0 \leq x \leq L$. Then $\phi(x)$ can be represented as:

$$\phi(x) = \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{L} \quad (39)$$

¶ How to choose the Fourier coefficients c_0, c_1, \dots in eq.(39)?

Exploit orthogonality:

$$\int_0^{\pi} \cos mx \cos nx \, dx = \begin{cases} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{cases} \quad (40)$$

To do this, multiply both sides of eq.(39) by $\cos \frac{k\pi x}{L}$ and integrate from 0 to L :

$$\int_0^L \phi(x) \cos \frac{k\pi x}{L} \, dx = \sum_{n=0}^{\infty} c_n \int_0^L \cos \frac{k\pi x}{L} \cos \frac{n\pi x}{L} \, dx \quad (41)$$

$$= \frac{c_k L}{2} \quad (42)$$

So, if we know the function $\phi(x)$ we can calculate the Fourier coefficients of its expansion:

$$c_k = \frac{2}{L} \int_0^L \phi(x) \cos \frac{k\pi x}{L} \, dx \quad (43)$$

¶ These Fourier coefficients have many interesting and important properties. First of all, they minimize the mean squared error between $\phi(x)$ and its expansion. That is, the c_n minimize:

$$S^2 = \int_0^L \left(\phi(x) - \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{L} \right)^2 \, dx \quad (44)$$

In fact,

$$\lim_{N \rightarrow \infty} S^2 = 0 \quad (45)$$

Another important property relates to truncated expansions:

$$\phi(x) = \sum_{n=0}^N c_n \cos \frac{n\pi x}{L} dx \quad (46)$$

Regardless of the order of the expansion, N :

- Orthogonality yields the same Fourier coefficients, c_k .
- These coefficients minimize the mean squared error of the truncated expansion.

¶ Band-limited function:

$$\phi(x) = \sum_{n=n_1}^{n_2} c_n \cos \frac{n\pi x}{L} \quad (47)$$

$$= c^T \gamma(x) \quad (48)$$

¶ Uncertainty in $\phi(x)$ is represented as uncertainty in Fourier coefficients c .

- For instance: c in ellipsoid of known shape and unknown size:

$$\mathcal{U}(h, \tilde{c}) = \left\{ \phi(x) = c^T \gamma(x) : (c - \tilde{c})^T W (c - \tilde{c}) \leq h^2 \right\}, \quad h \geq 0 \quad (49)$$

2.3 Geometry of Ellipsoids

¶ **Motivation:**

- Suppose we have limited 2-dimensional data about an uncertain phenomenon:

$$(c_1, c_2)_i, \quad i = 1, \dots, n \tag{50}$$

- These data, when plotted, spread over an ellipse-like cluster around (0,0).
- Future data might extend beyond this cluster.
- How to represent our uncertainty?

¶ **Preliminary question:**

- Consider the $c_1 \times c_2$ plane.
- What shape is described by: $c_1^2 + c_2^2 = h^2$? Circle.
- What shape is described by: $ac_1^2 + bc_2^2 = h^2$, where $a, b > 0$? Ellipse.
- What shape is described by: $ac_1^2 + gc_1c_2 + bc_2^2 = h^2$, where $a, b > 0$? Ellipse if the coefficients define a positive definite matrix.

¶ We need one more digression before we proceed with our example: Geometry of ellipsoids.

The question we study in this subsection is:

What are the **directions and lengths** of the principal axes of an ellipsoid?

¶ If: c is an N -vector and W is a real, symmetric, positive definite matrix, then an ellipsoid of c -vectors of dimension N is defined by:

$$c^T W c = h^2 \tag{51}$$

where h is any positive real number.

¶ Simple examples:

$$h^2 = c_1^2 w_1 + c_2^2 w_2, \quad W = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}, \quad w_i > 0 \tag{52}$$

$$h^2 = 2c_1^2 + c_1c_2 + 2c_2^2, \quad W = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{53}$$

¶ To answer our question, we must solve an optimization problem.

We must find vectors c which have two properties:

- Length is extremal.
- Lie on the boundary of the ellipsoid.

¶ To optimize the length of c , it is sufficient to optimize the square of the length of c .

So we must optimize:

$$c^T c \tag{54}$$

Let's try differential calculus:

$$0 = \frac{dc^T c}{dc} = 2c \implies c = 0 \tag{55}$$

That's the minimum. What's the maximum? $c^T c$ is unbounded. We need the constraint.

¶ To solve this problem we will use the method of **Lagrange multipliers**.

¶ A c -vector lies on the ellipsoid if eq.(51) is satisfied.

Expressing this slightly differently, the constraint on c is:

$$h^2 - c^T W c = 0 \quad (56)$$

¶ Define the objective function:

$$H = c^T c - \lambda(h^2 - c^T W c) \quad (57)$$

If we find all c -vectors which optimize H subject to the constraint, we will have solved the problem.

¶ Condition for extremum of H :

$$0 = \frac{\partial H}{\partial c} = 2c - 2\lambda W c \quad (58)$$

$$\implies (I - \lambda W)c = 0 \quad (59)$$

which means that:

c is an eigenvector of W .

$\frac{1}{\lambda}$ = the corresponding eigenvalue.

¶ Define the eigenvalues and orthonormal eigenvectors of W :

$$W v_i = \mu_i v_i, \quad i = 1, \dots, N \quad (60)$$

where:

$$0 < \mu_1 \leq \dots \leq \mu_N \quad \text{and} \quad v_m^T v_n = \delta_{mn} \quad (61)$$

where δ_{mn} is the Kronecker delta function:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad (62)$$

¶ Now, since c must be an eigenvector of W , we know that:

$$c = r v_i \quad (63)$$

for some non-zero r and for any $i = 1, \dots, N$.

Hence the constraint on c is:

$$h^2 = c^T W c = r^2 v_i^T W v_i = r^2 \mu_i \implies r = \pm \frac{h}{\sqrt{\mu_i}} \quad (64)$$

¶ Thus the optimizing c -vectors are:

$$c = \pm \frac{h}{\sqrt{\mu_i}} v_i, \quad i = 1, \dots, N \quad (65)$$

From this we see that:

The **directions** of the principal semi-axes are:

$$\pm v_1, \dots, \pm v_N \quad (66)$$

The **lengths** of the principal semi-axes are:

$$\frac{h}{\sqrt{\mu_1}}, \dots, \frac{h}{\sqrt{\mu_N}} \quad (67)$$

2.4 Fourier Ellipsoid Bounded Uncertain Load

Based on *Robust Reliability in the Mechanical Sciences*, section 3.2.4.

¶ We now consider a different type of prior information about the uncertain load profile $\phi(x)$.

¶ About $\phi(x)$ we **know**:

- Load vanishes at ends: $\phi(0) = \phi(L) = 0$.
- $\phi(x)$ is constrained to specific known spatial frequencies.
- The amplitudes of these frequencies are bounded by an ellipsoid of known shape.

¶ About $\phi(x)$ we **do not know**:

- The precise amplitudes of the Fourier coefficients.
- The size of the ellipsoid.

¶ In other words, a load profile is represented by:

$$\phi(x) = \sum_{n=n_1}^{n_2} c_n \sin \frac{n\pi x}{L} \tag{68}$$

$$= c^T \sigma(x) \tag{69}$$

where:

c = vector of unknown Fourier coefficients.

$\sigma(x)$ = vector of known corresponding sine functions.

¶ The uncertainty in $\phi(x)$ is represented by the following Fourier ellipsoid bound info-gap model:

$$\mathcal{U}(h, 0) = \left\{ \phi(x) = c^T \sigma : c^T W c \leq h^2 \right\}, \quad h \geq 0 \tag{70}$$

where W is a known, real, symmetric, positive definite matrix.

¶ The system model is obtained by combining eq.(4) on p.4 for the bending moment with eq.(69):

$$M(x) = c^T \left[\underbrace{-\frac{L-x}{L} \int_0^x u \sigma(u) \, du - \frac{x}{L} \int_x^L (L-u) \sigma(u) \, du}_{\zeta(x)} \right] \tag{71}$$

$$= c^T \zeta(x) \tag{72}$$

¶ As before, failure occurs if the bending moment exceeds a critical value, as expressed in eq.(5) on p.4.

For an example of a Fourier ellipsoid model see: Yakov Ben-Haim and Isaac Elishakoff, Non-Probabilistic models of uncertainty in the non-linear buckling of shells with general imperfections: Theoretical estimates of the knockdown factor. *A.S.M.E. Journal of Applied Mechanics*, Vol. 56, pp 403–410, 1989.

¶ In order to find the robustness, eq.(9), p.4, we must solve the following optimization:

$$\max M(x) \quad \text{for } c^T W c \leq h^2 \quad (73)$$

which is equivalent to:

$$\max c^T \zeta \quad \text{for } c^T W c \leq h^2 \quad (74)$$

To do this we employ the Cauchy inequality:

$$(x^T y)^2 \leq (x^T x) (y^T y) \quad (75)$$

with equality iff:

$$x \propto y \quad (76)$$

Let us write:

$$c^T \zeta = (W^{1/2} c)^T (W^{-1/2} \zeta) \quad (77)$$

Applying Cauchy's inequality to the expression on the right:

$$(c^T \zeta)^2 \leq \left[(W^{1/2} c)^T (W^{1/2} c) \right] \left[(W^{-1/2} \zeta)^T (W^{-1/2} \zeta) \right] \quad (78)$$

$$= \underbrace{[c^T W c]}_{\leq h^2} [\zeta^T W^{-1} \zeta] \quad (79)$$

From this we conclude that:

$$\max_{c \in \mathcal{U}(h,0)} M(x) = h \sqrt{\zeta(x)^T W^{-1} \zeta(x)} \quad (80)$$

¶ We can now express the robustness as the greatest value of the uncertainty parameter h at which the bending moment does not exceed the critical value. We find:

$$\hat{h} = \frac{M_c}{\max_{0 \leq x \leq L} \sqrt{\zeta(x)^T W^{-1} \zeta(x)}} \quad (81)$$

¶ Let us consider a **special case**:

W is the identity matrix, so the uncertainty ellipsoid is a sphere.

¶ Now $\zeta^T W \zeta$ becomes:

$$\zeta^T(x) \zeta(x) = \frac{L^4}{\pi^4} \sum_{n=n_1}^{n_2} \frac{1}{n^4} \sin^2 \frac{n\pi x}{L} \quad (82)$$

The terms in this sum decrease rapidly with n .

Hence the maximum is dominated by the first term:

$$\max_{0 \leq x \leq L} \sqrt{\zeta(x)^T \zeta(x)} \approx \max_{0 \leq x \leq L} \sqrt{\frac{L^4}{\pi^4} \frac{1}{n_1^4} \sin^2 \frac{n_1 \pi x}{L}} \quad (83)$$

$$= \frac{L^2}{n_1^2 \pi^2} \quad (84)$$

From eq.(81) we find the robustness to be:

$$\hat{h} \approx \frac{n_1^2 \pi^2 M_c}{L^2} \quad (85)$$

¶ Comparing this with the robustness for the uniform-bound info-gap model, with $\tilde{\phi} = 0$, eq.(15) on p.5:

$$\hat{h} = \frac{8M_c}{L^2} \quad (86)$$

we see that the reliability is substantially enhanced by constraining the spatial modes of the load function.

3 Conclusion

§ 3 components of reliability analysis:

1. A system model.
2. A failure criterion.
3. An uncertainty model.

§ Robustness:

- Maximum tolerable uncertainty.
- Basis for design selection.
- Combination of the 3 components.