## Lecture 3

# Probabilistic Reliability

## with

# Info-Gap Uncertainty

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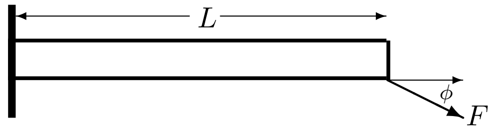
# Contents

# 1 Highlights

§ **Info-Gap Robustness Analysis of:**

- Random Loads on a Beam.
- Random Events and Failure.

# 2 Random Load on a Cantilever: Info-Gap Robustness Analysis




## 2.1 Problem Statement

- Rigid beam.
- $F =$ load at free end at angle $\phi$.
- $k =$ rotational stiffness at base.
- $\theta =$ angular rotation of beam:

$$\theta = \frac{F \sin\phi}{k} \tag{1}$$

- Design requirement:

$$|\theta| \le \theta_{\rm c} \tag{2}$$

- Problem: Load uncertain, $F$.

## 2.2 Uniform-Bound Info-Gap Model

§ **We know:**
- $F$ is nominally zero.
- $F$ may deviate greatly from zero.

§ **We do not know:**
- Maximum deviation from zero.
- Probability distribution of $F$.

§ **Info-gap model** of uncertainty in $F$:

$$\mathcal{U}(h) = \left\{ F : \ |F| \le h \right\}, \quad h \ge 0 \tag{3}$$

Two levels of uncertainty:
∘ $F$ unknown.
∘ Horizon of uncertainty, $h$, unknown.

§ **Derive the robustness** by combining:
- System model: eq.(1).
- Performance requirement: eq.(2).
- Uncertainty model: eq.(3).

$$\widehat{h}(\theta_{\rm c}) = \max \left\{ h : \ \left( \max_{F \in \mathcal{U}(h)} |\theta| \right) \le \theta_{\rm c} \right\} \tag{4}$$

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Based on problem 71 in ps2-02.tex.

§ **Solution method.** Start from the inside:

Let $m(h)$ denote the inner maximum in eq.(4) that occurs for $F = \pm h$:

$$m(h) = \left| \frac{h \sin\phi}{k} \right| \le \theta_{\rm c} \quad \Longrightarrow \quad \boxed{\widehat{h}(\theta_{\rm c}) = \frac{k\theta_{\rm c}}{\sin\phi}} \tag{5}$$

§ **Two properties** of all info-gap robustness functions, $\widehat{h}(\theta_{\rm c})$:

- **Trade off:** Better performance (smaller $\theta_{\rm c}$) has worse robustness (lower $\widehat{h}$).
- **Zeroing:** Predicted performance (no rotation) has zero robustness.

§ **Inverse of robustness:** $m(h)$ is the inverse function of $\widehat{h}(\theta_{\rm c})$:

$$m(h) = \theta_{\rm c} \quad \text{if and only if} \quad \widehat{h}(\theta_{\rm c}) = h \tag{6}$$

Hence: plot of $m(h)$ vs $h$ is the same as plot of $\theta_{\rm c}$ vs $\widehat{h}(\theta_{\rm c})$.

## 2.3 Fractional-Error Info-Gap Model

§ **Different information, different robustness.**

§ **We know:**

- $F$ nominally equals $\widetilde{F}$, a known positive value.
- $F$ may deviate greatly from $\widetilde{F}$.
- $k$ nominally equals $\widetilde{k}$, a known positive value.
- $k$ may deviate greatly from $\widetilde{k}$.
- $k$ is non-negative.

§ **We do not know:**

- Maximum fractional deviation of $F$ from $\widetilde{F}$, or of $k$ from $\widetilde{k}$.
- Probability distribution of $F$ or of $k$.

§ **Info-gap model** of uncertainty in $F$ and $k$:

$$\mathcal{U}(h) = \left\{ F, k : \ \left| \frac{F - \widetilde{F}}{\widetilde{F}} \right| \le h, \ k > 0, \ \left| \frac{k - \widetilde{k}}{\widetilde{k}} \right| \le h \right\}, \quad h \ge 0 \tag{7}$$

§ **Derive the robustness** by combining:

- System model: eq.(1), p.4: $\theta = (F \sin\phi)/k$.
- Performance requirement: eq.(2), p.4: $|\theta| \le \theta_{\rm c}$.
- Uncertainty model: eq.(7).

$$\widehat{h}(\theta_{\rm c}) = \max \left\{ h : \ \left( \max_{F, k \in \mathcal{U}(h)} |\theta| \right) \le \theta_{\rm c} \right\} \tag{8}$$

§ **Solution method:** start with the inner maximum of eq.(8).

The inner maximum, $m(h)$, occurs at:

$$F = (1+h)\widetilde{F}, \quad k = \max[0, \ (1-h)\widetilde{k}] \tag{9}$$

Thus, for $h < 1$:

$$m(h) = \frac{(1+h)\widetilde{F}\sin\phi}{(1-h)\widetilde{k}} \le \theta_{\rm c} \quad \Longrightarrow \quad (1+h)\widetilde{F}\sin\phi \le (1-h)\widetilde{k}\theta_{\rm c} \quad \Longrightarrow \quad \boxed{\widehat{h} = \frac{\widetilde{k}\theta_{\rm c} - \widetilde{F}\sin\phi}{\widetilde{k}\theta_{\rm c} + \widetilde{F}\sin\phi}} \tag{10}$$

or zero if this is negative. Note that $\widehat{h}$ is less than 1.

§ **Two properties:**
- Trade off: greater robustness only at greater allowed deflection.
- Zero robustness at estimated deflection.

§ **Meaning of numerical values of $\widehat{h}$:**
- $\widehat{h} = 0.2$ implies performance guaranteed up to 20% error in both $\widetilde{F}$ and $\widetilde{k}$.
- $\widehat{h} = 0.7$ implies performance guaranteed up to 70% error in both $\widetilde{F}$ and $\widetilde{k}$.
- Asymptotic robustness:

$$\lim_{\theta_{\rm c}\to\infty} \widehat{h}(\theta_{\rm c}) = 1 \tag{11}$$

- Max possible robustness (in this problem:) immunity to 100% error.
  - Small? Large? Large enough?
  - Important and difficult **value judgment.**

## 2.4 Probability of Failure

§ Different prior knowledge:
- $k$ is known.
- $F$ is exponentially distributed random variable:

$$p(F) = \lambda e^{-\lambda F}, \quad F \ge 0 \tag{12}$$

§ **Failure of failure:**
- **Mechanical** failure [violating design requirement, eq.(2)]:

$$|\theta| > \theta_{\rm c} \tag{13}$$

- **Probability** of failure:

$$P_{\rm f} = {\rm Prob}(|\theta| > \theta_{\rm c}) \tag{14}$$

§ **Deriving probability of failure:**

$F$ is non-negative so $\theta$ is also non-negative. Hence the probability of failure is:

$$P_{\rm f}(\lambda) = \text{Prob}(|\theta| > \theta_{\rm c}) = \text{Prob}(\theta > \theta_{\rm c}) = \text{Prob}\left(\frac{F\sin\phi}{k} > \theta_{\rm c}\right) = \text{Prob}\left(F > \frac{k\theta_{\rm c}}{\sin\phi}\right) = \boxed{\exp\left(-\frac{\lambda k\theta_{\rm c}}{\sin\phi}\right)} \tag{15}$$

## 2.5 Hybrid Uncertainty: Probability with Info-Gaps

§ Continue from section 2.4, but with $\lambda$ **uncertain**.

§ **We know:**
- $\widetilde{\lambda}$, an estimate of $\lambda$.
- $\lambda$ is positive.

§ **We do not know:**
- Maximum fractional error of the estimate.
- Probability distribution of $\lambda$.

§ **Info-gap model** for uncertainty in $\lambda$:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda > 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}} \right| \le h \right\}, \quad h \ge 0 \tag{16}$$

§ **Two types of failure:**
- **Mechanical failure.** Rotation too large:

$$|\theta| > \theta_{\rm c} \tag{17}$$

- **Probabilistic failure.** Probability of failure too large:

$$\mathrm{Prob}(|\theta| > \theta_{\rm c}) > P_{\rm c} \tag{18}$$

§ **Evaluate robustness** with respect to probabilistic failure:

$$\widehat{h} = \max \left\{ h : \ \left( \max_{\lambda \in \mathcal{U}(h)} P_{\rm f}(\lambda) \right) \le P_{\rm c} \right\} \tag{19}$$

- Start with the inner maximum of eq.(19), $m(h)$.
- From eq.(15), p.6, the inner maximum occurs at $\lambda = \max[0, \ (1-h)\widetilde{\lambda}]$:

$$m(h) = \exp\left( - \frac{(1-h)\widetilde{\lambda} k \theta_{\rm c}}{\sin\phi} \right) \le P_{\rm c} \quad \Longrightarrow \quad \frac{(1-h)\widetilde{\lambda} k \theta_{\rm c}}{\sin\phi} \ge -\ln P_{\rm c} \quad \Longrightarrow \quad \boxed{\widehat{h}(P_{\rm c}) = 1 + \frac{\sin\phi}{\widetilde{\lambda} k \theta_{\rm c}} \ln P_{\rm c}} \tag{20}$$

or zero if this is negative.

§ **Two properties:**
- **Trade off:** $\widehat{h}(P_{\rm c})$ decreases (gets worse) as $P_{\rm c}$ decreases (gets better).
- **Zeroing:** Robustness vanishes at nominal $P_{\rm f}$:

$$\widehat{h}(P_{\rm c}) = 0 \quad \text{if} \quad P_{\rm c} = P_{\rm f}(\widetilde{\lambda}) = \exp\left( - \frac{\widetilde{\lambda} k \theta_{\rm c}}{\sin\phi} \right) \tag{21}$$

# 3 Random Events and Failure: Info-Gap Robustness Analysis

## 3.1 Formulation

§ **Problem Statement:**
- **Adverse events occur** randomly, independently, with average rate $\lambda$/sec.
- **System fails** if $n$ or more events occur within time $T$.

§ **Questions:**
- What is probability of failure if $n = 1$ or $n = 2$?
- Suppose $\lambda$ is uncertain. Evaluate robustness of failure probability.

## 3.2 Probabilities of Failure

§ Adverse events occur according to a **Poisson process:**
- Independent random events, constant average rate.
- Probability of exactly $n$ events in duration $T$ is:

$$P_n(T) = \frac{(\lambda T)^n}{n!} \mathrm{e}^{-\lambda T}, \quad n = 0,\, 1,\, 2,\, \ldots \tag{22}$$

§ **Failure probability for** $n = 1$**:**
- The probability of **no** events up to time $T$ is $P_0(T)$.
- Thus, for $n = 1$, the probability of failure is $1 - P_0(T)$:

$$\boxed{P_{\mathrm{f},1} = 1 - \mathrm{e}^{-\lambda T}} \tag{23}$$

§ **Failure probability for** $n = 2$**:**
- The probability of less than 2 events up to time $T$ is $P_0(T) + P_1(T)$.
- Thus, for $n = 2$, the probability of failure is $1 - P_0(T) - P_1(T)$:

$$\boxed{P_{\mathrm{f},2} = 1 - \mathrm{e}^{-\lambda T} - \lambda T \mathrm{e}^{-\lambda T}} \tag{24}$$

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## 3.3 Uncertain Poisson Process

§ **We know:**
- $\widetilde{\lambda}$ = estimate of failure rate, $\lambda$.
- $s$ = estimate of error of $\widetilde{\lambda}$.
- $\lambda$ is positive.

§ **We do not know:**
- True value of $\lambda$.
- Maximum fractional error of estimate.
- Probability distribution for $\lambda$.

§ **Info-gap model** for uncertainty in $\lambda$:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda > 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{25}$$

§ **Two properties of all info-gap models:**
- **Contraction:**

$$\mathcal{U}(h) = \left\{ \widetilde{\lambda} \right\} \tag{26}$$

- **Nesting:**

$$h < h' \quad \Longrightarrow \quad \mathcal{U}(h) \subseteq \mathcal{U}(h') \tag{27}$$

## 3.4 Robustness to Info-Gap Uncertainty in Poisson Process

§ **System model:** $P_{\mathrm{f,n}}$ in eq.(23) or (24).

§ **Performance requirement.** Failure probability acceptably small:

$$P_{\mathrm{f,n}} \le P_{\mathrm{c}} \tag{28}$$

§ **Uncertainty model:** eq.(25).

§ **Robustness function combines** system model, performance requirement, and uncertainty model.

§ **Evaluating the robustness** for $n = 1$.
- The robustness is defined as:

$$\widehat{h}_1(P_{\mathrm{c}}) = \max \left\{ h : \ \left( \max_{\lambda \in \mathcal{U}(h)} P_{\mathrm{f,1}} \right) \le P_{\mathrm{c}} \right\} \tag{29}$$

- Let $m_1(h)$ denote the inner maximum of eq.(29).
- According to eq.(23), $m(h)$ occurs when $\lambda$ is as large as possible: $\lambda = \widetilde{\lambda} + sh$. Thus:

$$m_1(h) = 1 - \mathrm{e}^{-(\widetilde{\lambda}+sh)T} \le P_{\mathrm{c}} \quad \Longrightarrow \quad \boxed{\widehat{h}_1(P_{\mathrm{c}}) = \frac{-\widetilde{\lambda}T - \ln(1 - P_{\mathrm{c}})}{sT}} \tag{30}$$

or zero if this is negative.

- Note trade off and zeroing.

§ **Evaluating the inverse of the robustness** for $n = 2$.

- The robustness is defined as:

$$\widehat{h}_2 = \max\left\{h : \left(\max_{\lambda\in\mathcal{U}(h)} P_{\mathrm{f},2}\right) \le P_{\mathrm{c}}\right\} \tag{31}$$

- Let $m_2(h)$ denote the inner maximum of eq.(31), which is the **inverse of the robustness.**
- From eq.(24), p.8, we find:

$$\frac{\partial P_{\mathrm{f},2}}{\partial \lambda} = \lambda T^2 \mathrm{e}^{-\lambda T} > 0 \tag{32}$$

- Thus $m_2(h)$ occurs when $\lambda$ is as large as possible: $\lambda = \widetilde{\lambda} + sh$.
- Thus, from eq.(24):

$$m_2(h) = 1 - \mathrm{e}^{-(\widetilde{\lambda}+sh)T} - (\widetilde{\lambda} + sh)T\mathrm{e}^{-(\widetilde{\lambda}+sh)T} \tag{33}$$

- The robustness is the greatest $h$ at which:

$$m_2(h) \le P_{\mathrm{c}} \tag{34}$$

- **Problem:** We can't solve eq.(34) for $h$.
- **Solution:** No need to.
  - ◦ $m_2(h)$ is the inverse of $\widehat{h}(P_{\mathrm{c}})$.
  - ◦ Plot of $h$ vs $m_2(h)$ equivalent to plot of $\widehat{h}(P_{\mathrm{c}})$ vs $P_{\mathrm{c}}$.

# 4 Conclusion

## In Conclusion

§ Info-gap uncertainty:

innovation, discovery, ignorance, surprise.

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## In Conclusion

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§ Optimism: our models get better all the time.

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§ Responsible decision making:

- Specify your goals.
- Maximize your robustness to uncertainty.
- Study the trade offs.
- Exploit windfall opportunities.