Lecture 4

Vibration Suppression

with

Uncertain Load

∼Info-Gap Analysis∼

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1 Design of a Vibrating Cantilever

1.1 Design Problem

¶ We now consider an example:
Vibration control in a cantilever subject to uncertain dynamic excitation.

¶ The cantilever: rigid beam which is clamped at one end.
See transparency of: • Galileo's cantilever.
• Atomic force microscope.

¶ The cantilever is the paradigm for:
- Tall building.
- Radio tower.
- Crane (agoran).
- Airplane wing.
- Turbine blade.
- Diving board.
- Canon barrel.
- Atomic force microscope.
- etc.

¶ Central goal in design of the cantilever:
Control of vibration resulting from external loads.

¶ Two basic approaches:
1. Prevent vibration by stiffening the beam.
2. Absorb vibration by dissipating energy.

¶ These design concepts are **not** mutually exclusive.
They can be implemented together.
¶ These design concepts are relevant in different circumstances as we will see.

Yakov Ben-Haim, *Info-Gap Decision Theory,* 2nd ed., section 3.3.1. See also lecture notes ro02.tex, section 6.

1.2 Robustness Function

¶ We will use the **robustness function** to evaluate the design options.

¶ Later we will consider the **opportuneness function.**

¶ As usual, the three components of the analysis are:
1. System model.
2. Failure (or performance) criterion.
3. Uncertainty model.

¶ We use a simple **system model:**
Rigid vibration around the clamped base.
$\theta(t) =$ angle of deflection of beam [radian].
$u(t) =$ moment of force at base, [Nm].
Equation of motion:

$$J\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + k\theta = u(t) \tag{1}$$

$J =$ moment of inertia of beam wrt rotation at base, $\int_0^L m(x)x^2\,\mathrm{d}x$.
$c =$ damping coefficient.
$k =$ rotational stiffness coefficient, [Nm/radian].

¶ Solution of eq. of motion, for:
- Zero initial conditions, $\theta(0) = \dot{\theta}(0) = 0$
- Subcritical damping, $\zeta^2 < 1$:

$$\theta_u(t) = \int_0^t u(\tau)f(t-\tau)\,\mathrm{d}\tau \tag{2}$$

$f(t) =$ impulse response function:

$$f(t) = \frac{1}{J\omega_\mathrm{d}}\mathrm{e}^{-\zeta\omega t}\sin\omega_\mathrm{d}t \tag{3}$$

$\omega^2 = k/J =$ squared natural frequency.
$\zeta = \frac{c}{2J\omega} =$ dimensionless damping coefficient.
$\omega_\mathrm{d} = \omega\sqrt{1-\zeta^2} =$ damped natural frequency.

¶ We now consider the **uncertainty model.**

What we **know** about the load is:

- The nominal load, $\widetilde{u}(t)$.
- The actual loads are transient:
  - May vary rapidly,
  - May attain large deviations from the nominal load.
  - No sustained deviation from the nominal load

We will model load uncertainty with the **cumulative energy bound** info-gap model:

$$\mathcal{U}(h,\widetilde{u}) = \left\{ u(t): \int_0^\infty \left[u(t) - \widetilde{u}(t) \right]^2 \, \mathrm{d}t \le h^2 \right\}, \quad h \ge 0 \tag{4}$$

¶ The **performance criterion:** Deflection must not exceed critical value:

$$|\theta(t)| \le \theta_{\mathrm{c}} \tag{5}$$

In terms of reward functions, define:

$$R(q,u) = |\theta(t)| \tag{6}$$

$u =$ uncertain load.
$q =$ design concept, as expressed in damping c and stiffness k.

¶ The robustness function can be defined as:

$$\widehat{h}(q,\theta_{\mathrm{c}}) = \max \left\{ h: \ \left(\max_{u \in \mathcal{U}(h,\widetilde{u})} |\theta_u(t)| \right) \le \theta_{\mathrm{c}} \right\} \tag{7}$$

$\widehat{h}(q,\theta_{\mathrm{c}})$ is the maximum tolerable info-gap.

¶ We now evaluate:

$$\max_{u \in \mathcal{U}(h,\widetilde{u})} |\theta_u(t)| \tag{8}$$

¶ Note that $\theta_u(t)$ in eq.(2) on p.4 can be re-written:

$$\theta_u(t) = \int_0^t u(\tau) f(t-\tau) \, \mathrm{d}\tau \tag{9}$$

$$= \int_0^t \left[u(\tau) - \widetilde{u}(\tau) \right] f(t-\tau) \, \mathrm{d}\tau + \underbrace{\int_0^t \widetilde{u}(\tau) f(t-\tau) \, \mathrm{d}\tau}_{\widetilde{\theta}(t)} \tag{10}$$

where $\widetilde{\theta}(t) =$ nominal deflection.

We need the Schwarz inequality:

$$\left(\int_a^b f(t)g(t)\,\mathrm{d}t\right)^2 \le \int_a^b f(t)^2\,\mathrm{d}t \int_a^b g(t)^2\,\mathrm{d}t \tag{11}$$

with equality iff:

$$f(t) = cg(t) \tag{12}$$

for any non-zero constant c.

Now notice that the first integral in eq.(10) on p.5 is bounded:

$$\left(\int_0^t \left[u(\tau)-\widetilde{u}(\tau)\right] f(t-\tau)\,\mathrm{d}\tau\right)^2 \le \underbrace{\left(\int_0^t \left[u(\tau)-\widetilde{u}(\tau)\right]^2\,\mathrm{d}\tau\right)}_{\text{I}} \underbrace{\left(\int_0^t f^2(t-\tau)\,\mathrm{d}\tau\right)}_{\text{II}} \tag{13}$$

¶ Note:

- From the info-gap model we know that: Integral I $\le h^2$.
- Integral II is known.
- The info-gap model allows us to choose $u(\tau)$ such that:

$$u(\tau) - \widetilde{u}(\tau) \propto f(t-\tau) \tag{14}$$

- Thus, from eqs.(10) and (13):

$$\max_{u\in\mathcal{U}(h,\widetilde{u})} |\theta_u(t)| = h\sqrt{\int_0^t f^2(\tau)\,\mathrm{d}\tau} + \left|\widetilde{\theta}(t)\right| \tag{15}$$

¶ We can now express the robustness function:

- Equate $\max|\theta_u(t)|$ to θ_{c}.
- Solve for h, yielding $\widehat{h}$:

$$h\sqrt{\int_0^t f^2(\tau)\,\mathrm{d}\tau} + \left|\widetilde{\theta}(t)\right| = \theta_{\mathrm{c}} \quad\Longrightarrow\quad \widehat{h}(q,\theta_{\mathrm{c}}) = \frac{\theta_{\mathrm{c}} - \left|\widetilde{\theta}(t)\right|}{\sqrt{\int_0^t f^2(\tau)\,\mathrm{d}\tau}} \tag{16}$$

unless this is negative, in which case $\widehat{h} = 0$.

1.3 Numerical Example

¶ We will consider a specific example. Nominal input $\widetilde{u}(t)$ is square:

$$\widetilde{u}(t) = \left\{ \begin{array}{lr} \widetilde{u}_o, & 0 \le t \le T \\ 0, & t > T \end{array} \right. \tag{17}$$

The nominal response can be calculated:

$$\widetilde{\theta}(t) = \theta_{\widetilde{u}}(t) = \frac{(1-\zeta^2)\widetilde{u}_o}{J\omega_{\rm d}}\gamma(t) \tag{18}$$

where $\gamma(t)$ is a known function.
For notational convenience we represent integral II in eq.(13) on p.6 as:

$$\sqrt{\int_0^t f^2(t-\tau)\,{\rm d}\tau} = \frac{1-\zeta^2}{2J\omega_{\rm d}^{3/2}}\phi(t) \tag{19}$$

where $\phi(t)$ is a known function.

Now the robustness function can be expressed:

$$\widehat{h}(q,\theta_{\rm c}) = \frac{2J\theta_{\rm c}\omega^2\sqrt{\omega_{\rm d}} - 2\sqrt{\omega_{\rm d}}|\widetilde{u}_o\gamma(t)|}{\omega\phi(t)} \tag{20}$$

Recall: $q =$ decision vector = $(c,\, k)$, which is embedded in ω and $\omega_{\rm d}$.

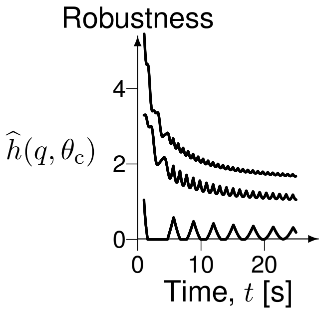

Figure 1: Robustness versus time for three values of the natural frequency $\omega = 1$, 3 and 4 (bottom to top). Negligible damping: $\zeta = 0.01$. $1 = J\theta_{\rm c} = \widetilde{u}_0$. $T = 5$.

¶ $\widehat{h}(q,\theta_{\rm c})$ vs. t is plotted in fig. 1
For various natural frequencies: $\omega =$ 1, 3 and 4 (bottom to top).
With negligible damping: $\zeta = 0.01$.

- $\widehat{h}$ oscillates but tends to decrease over time.
- At low stiffness ($\omega = 1$) the robustness periodically vanishes.
- At moderate and high stiffness ($\omega = 3$, 4)
  $\widehat{h}$ oscillates but does not reach zero for the duration shown.
- The transition from rapid to slow decrease in $\widehat{h}$
  occurs about at $t = T$ (end of nominal input).

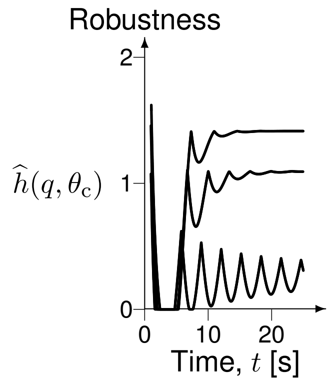

Figure 2: Robustness versus time for three values of the damping ratio $\zeta =$ 0.03, 0.3, 0.5 (bottom to top). Fixed natural frequency $\omega =$ 1. $1 = J\theta_c = \widetilde{\mathcal{U}}_0$. $T = 5$.

¶ Now consider fig. 2, which shows
$\widehat{h}(q, \theta_c)$ vs. t for various damping ratios:
$\zeta =$ 0.03, 0.3 and 0.5
at low stiffness: $\omega = 1$.

- Lowest curve is quite similar to lowest curve in fig. 1.
- With large damping ($\zeta =$ 0.3 or 0.5):
  $\widehat{h}$ is small for $t \leq T$
  $\widehat{h}$ is large and nearly constant thereafter.

¶ Comparing figs. 1 and 2:
- Fig. 1 is based on the "stiffness" design concept, with negligible damping.
- Fig. 2 is based on the "dissipation" design concept, with negligible stiffness.
- The choice of a design concept depends on the time frame of interest:
  - ◦ $t < T$ calls for "stiffness" design.
  - ◦ $t > T$ calls for "dissipation" design.
  - ◦ $t > 0$ calls for combined "stiffness" and "dissipation" design.

1.4 Opportuneness Function

¶ We now consider the opportuneness function.

Windfall reward: angular deflection $\theta_{\rm w}$ **much less** (much better) than the survival requirement, $\theta_{\rm c}$:

$$\theta_{\rm w} < \widetilde{\theta} < \theta_{\rm c} \tag{21}$$

¶ Immunity to windfall, $\widehat{\beta}(q, \theta_{\rm w})$: the **least** info-gap at which windfall is **possible.**

¶ Analogous to eq.(7) on p. 5:

$$\widehat{\beta}(q, \theta_{\rm w}) = \min \left\{ h : \min_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| \le \theta_{\rm w} \right\} \tag{22}$$

¶ **Smaller is better** for $\widehat{\beta}$. Unlike $\widehat{h}$, for which **bigger is better.**

¶ Proceeding as in eq.(15) on p. 6 we find:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| = -h \sqrt{\int_0^t f^2(\tau) \, {\rm d}\tau} + \left| \widetilde{\theta}(t) \right| \tag{23}$$

Equating this to $\theta_{\rm w}$ and solving for h yields the opportuneness function, as in eq.(16) on p. 6:

$$-h \sqrt{\int_0^t f^2(\tau) \, {\rm d}\tau} + \left| \widetilde{\theta}(t) \right| = \theta_{\rm w} \quad \Longrightarrow \quad \widehat{\beta}(q, \theta_{\rm w}) = \frac{\left| \widetilde{\theta}(t) \right| - \theta_{\rm w}}{\sqrt{\int_0^t f^2(\tau) \, {\rm d}\tau}} \tag{24}$$

unless this is negative, in which case $\widehat{\beta} = 0$.

Why does $\widehat{\beta} = 0$ in this case?

$\widehat{\beta} < 0$ only if $\left| \widetilde{\theta}(t) \right| < \theta_{\rm w}$.
This means that the **nominal response** $|\widetilde{\theta}(t)|$
is less than the **windfall response** $\theta_{\rm w}$.
Hence windfall is possible even without uncertainty: The immunity to windfall is zero.

¶ Compare $\widehat{\beta}(q,\theta_{\rm w})$ to the robustness in eq.(16) on p. 6:

$$\widehat{h}(q,\theta_{\rm c}) = \frac{\theta_{\rm c} - \left|\widetilde{\theta}(t)\right|}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}} \tag{25}$$

We see that the immunity functions are related as:

$$\widehat{\beta}(q,\theta_{\rm w}) = -\widehat{h}(q,\theta_{\rm c}) + \frac{\theta_{\rm c} - \theta_{\rm w}}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}} \tag{26}$$

¶ We now consider **antagonism** and **sympathy** of the immunity functions.

¶ The immunity functions $\widehat{\beta}(q,\theta_{\rm w})$ and $\widehat{h}(q,\theta_{\rm c})$ are **sympathetic** if they can be improved simultaneously.
They are **antagonistic** if either can be improved only at the expense of the other.

¶ For example, we can vary ω. The immunity functions are **antagonistic** if:

$$\underbrace{\frac{\partial\widehat{h}(q,\theta_{\rm c})}{\partial\omega} > 0}_{\text{improving with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial\widehat{\beta}(q,\theta_{\rm w})}{\partial\omega} > 0}_{\text{degenerating with } \omega} \tag{27}$$

or if:

$$\underbrace{\frac{\partial\widehat{h}(q,\theta_{\rm c})}{\partial\omega} < 0}_{\text{degenerating with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial\widehat{\beta}(q,\theta_{\rm w})}{\partial\omega} < 0}_{\text{improving with } \omega} \tag{28}$$

¶ On the other hand, the immunity functions are **sympathetic** if:

$$\underbrace{\frac{\partial\widehat{h}(q,\theta_{\rm c})}{\partial\omega} > 0}_{\text{improving with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial\widehat{\beta}(q,\theta_{\rm w})}{\partial\omega} < 0}_{\text{improving with } \omega} \tag{29}$$

or if:

$$\underbrace{\frac{\partial\widehat{h}(q,\theta_{\rm c})}{\partial\omega} < 0}_{\text{degenerating with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial\widehat{\beta}(q,\theta_{\rm w})}{\partial\omega} > 0}_{\text{degenerating with } \omega} \tag{30}$$

¶ In short, the immunity functions are **sympathetic** wrt ω if and only if:

$$\frac{\partial\widehat{h}(q,\theta_{\rm c})}{\partial\omega}\frac{\partial\widehat{\beta}(q,\theta_{\rm w})}{\partial\omega} < 0 \tag{31}$$

¶ Return to eq.(26) on p. 10.

- Question: Under what conditions will $\widehat{h}$ and $\widehat{\beta}$ always be sympathetic?
- Answer: If and only if their optima coincide. See fig. 3.

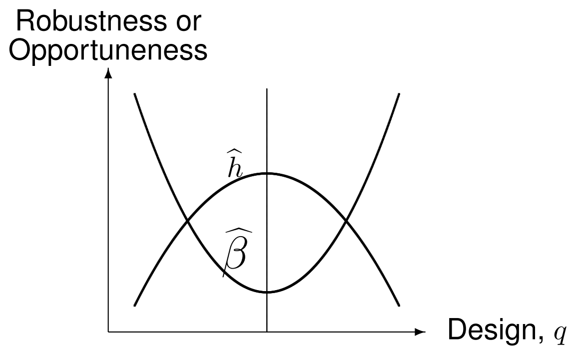

Figure 3: Sympathetic robustness and opportuneness curves.

¶ When will this occur? Iff

$$\frac{\partial\widehat{\beta}}{\partial q} = 0 = \frac{\partial\widehat{h}}{\partial q} \tag{32}$$

From eq.(26) we see that this will happen only if, at the same q, we also have:

$$\frac{\partial D}{\partial q} = 0 \tag{33}$$

where we define:

$$D = \frac{\theta_{\rm c} - \theta_{\rm w}}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}} \tag{34}$$

"Usually" this will not happen, which means that, instead of fig. 3, we will have fig. 4.

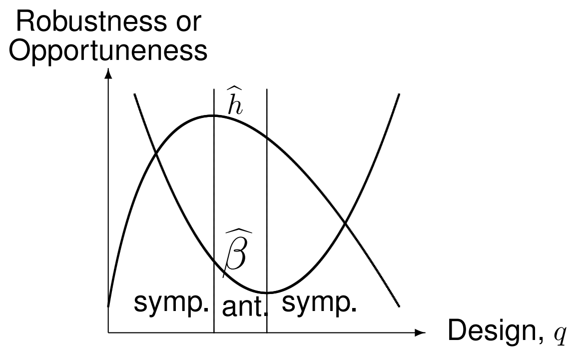

Figure 4: Robustness and opportuneness curves which are both sympathetic and antagonistic.