55. **Adaptive force balancing.** (p.235) A downward distributed load is applied on a straight unit interval. Denote the load $L(x)$ for $0 \le x \le 1$. Uncertainty in the load is described by:

$$\mathcal{U}(h) = \left\{ L(x) : \left| \frac{L(x) - \widetilde{L}}{\widetilde{L}} \right| \le h \right\}, \quad h \ge 0 \tag{183}$$

where $\widetilde{L}$ is known and positive. The designer must choose a distributed restoring force directed upward along the same unit interval. Denote the restoring force $R(x)$ for $0 \le x \le 1$. We require that the net moment of force around $x = 0$ not exceed the critical value M_c. Construct the robustness function for each of the following designs, and discuss your preferences among the designs:

(a) Designer 1 suggests choosing $R(x) = \widetilde{L}$.

(b) Designer 2 suggests an adaptive procedure whereby the restoring force is constant along the interval, and equal to the average of the actually realized force: $R(x) = \int_0^1 L(y)\,dy$.

(c) Designer 3 suggests an adaptive procedure whereby the restoring force is constant along the interval, and equal to the average of the actually realized force: $R(x) = \int_0^1 L(y)\,dy$. However, the adaptive procedure introduces additional uncertainty to the load, so eq.(183) is replaced by:

$$\mathcal{U}(h) = \left\{ L(x) : \left| \frac{L(x) - \widetilde{L}}{w\widetilde{L}} \right| \le h \right\}, \quad h \ge 0 \tag{184}$$

where $w > 1$ and known.

(d) Designer 4 suggests an adaptive procedure whereby the restoring force is linearly increasing along the interval, and equal at the midpoint to the average of the actually realized force: $R(x) = 2x \int_0^1 L(y)\,dy$.

(e) Designer 5 suggests an adaptive procedure whereby the restoring force is linearly decreasing along the interval, and equal at the midpoint to the average of the actually realized force: $R(x) = 2(1-x) \int_0^1 L(y)\,dy$.

Solution for problem 55: Adaptive force balancing. (p.54)

The robustness of design R with requirement M_c is:

$$\widehat{h}(R, M_c) = \max \left\{ h : \left(\max_{L \in \mathcal{U}(h)} \int_0^1 x[L(x) - R(x)] \, dx \right) \le M_c \right\} \tag{1385}$$

Let $\mu(h)$ denote the inner maximum, which is the inverse of the robustness function.

(a) Designer 1: $R(x) = \widetilde{L}$. $\mu(h)$ occurs for $L(x) = (1+h)\widetilde{L}$:

$$\mu(h) = \int_0^1 x[(1+h)\widetilde{L} - \widetilde{L}] \, dx = h\widetilde{L} \int_0^1 x \, dx = \frac{h\widetilde{L}}{2} \tag{1386}$$

Equate this to M_c and solve for h to obtain the robustness:

$$\boxed{\widehat{h}_1(\widetilde{L}, M_c) = \frac{2M_c}{\widetilde{L}}} \tag{1387}$$

(b) Designer 2: $R(x) = \int_0^1 L(y) \, dy$.

$$\begin{aligned} \mu(h) &= \max_{L \in \mathcal{U}(h)} \int_0^1 x[L(x) - R(x)] \, dx && (1388) \\ &= \max_{L \in \mathcal{U}(h)} \left[\int_0^1 xL(x) \, dx - \int_0^1 x \, dx \int_0^1 L(y) \, dy \right] && (1389) \\ &= \max_{L \in \mathcal{U}(h)} \int_0^1 \left[\left(x - \frac{1}{2} \right) L(x) \, dx \right] && (1390) \end{aligned}$$

$x - \frac{1}{2}$ is negative for $x < \frac{1}{2}$ and positive otherwise. Thus the maximum occurs for:

$$L(x) = \begin{cases} (1-h)\widetilde{L} & \text{if } x < \frac{1}{2} \\ (1+h)\widetilde{L} & \text{else} \end{cases} \tag{1391}$$

Thus:

$$\mu(h) = (1-h)\widetilde{L} \int_0^{1/2} \left(x - \frac{1}{2} \right) \widetilde{L} \, dx + (1+h)\widetilde{L} \int_{1/2}^1 \left(x - \frac{1}{2} \right) \widetilde{L} \, dx = \cdots = \frac{\widetilde{L}h}{4} \tag{1392}$$

Equate this to M_c and solve for h to obtain the robustness:

$$\boxed{\widehat{h}_2(\widetilde{L}, M_c) = \frac{4M_c}{\widetilde{L}}} \tag{1393}$$

Comparing with eq.(1387) we see that the adaptive design (designer 2) is twice as robust as the nominal optimal design (designer 1).

(c) Designer 3: $R(x) = \int_0^1 L(y) \, dy$ with added uncertainty, eq.(184). The solution is the same as in part (b) except that now eq.(1391) becomes:

$$L(x) = \begin{cases} (1-wh)\widetilde{L} & \text{if } x < \frac{1}{2} \\ (1+wh)\widetilde{L} & \text{else} \end{cases} \tag{1394}$$

Thus eq.(1392) becomes:

$$\mu(h) = (1-wh)\widetilde{L} \int_0^{1/2} \left(x - \frac{1}{2} \right) dx + (1+wh)\widetilde{L} \int_{1/2}^1 \left(x - \frac{1}{2} \right) dx = \cdots = \frac{w\widetilde{L}h}{4} \tag{1395}$$

Equate this to M_c and solve for h to obtain the robustness:

$$\boxed{\widehat{h}_3(\widetilde{L}, M_c) = \frac{4M_c}{w\widetilde{L}}} \tag{1396}$$

Comparing this to eq.(1393) shows the decrement of robustness resulting from the added uncertainty.

(d) Designer 4: $R(x) = 2x \int_0^1 L(y)\,dy$. (Note: $\int_0^1 x \times 2x\,dx = 2/3$.)

$$\begin{aligned} \mu(h) &= \max_{L\in\mathcal{U}(h)} \int_0^1 x[L(x) - R(x)]\,dx & (1397)\\ &= \max_{L\in\mathcal{U}(h)} \left[\int_0^1 xL(x)\,dx - \int_0^1 x \times 2x\,dx \int_0^1 L(y)\,dy\right] & (1398)\\ &= \max_{L\in\mathcal{U}(h)} \int_0^1 \left[\left(x - \frac{2}{3}\right) L(x)\,dx\right] & (1399) \end{aligned}$$

$x - \frac{2}{3}$ is negative for $x < \frac{2}{3}$ and positive otherwise. Thus the maximum occurs for:

$$L(x) = \begin{cases} (1-h)\widetilde{L} & \text{if } x < \frac{2}{3} \\ (1+h)\widetilde{L} & \text{else} \end{cases} \tag{1400}$$

Thus:

$$\mu(h) = (1-h)\widetilde{L} \int_0^{2/3} \left(x - \frac{2}{3}\right) \widetilde{L}\,dx + (1+h)\widetilde{L} \int_{2/3}^1 \left(x - \frac{2}{3}\right) \widetilde{L}\,dx = \cdots = -\frac{3\widetilde{L}}{18} + \frac{5\widetilde{L}h}{18} \tag{1401}$$

Equate this to M_c and solve for h to obtain the robustness:

$$\boxed{\widehat{h}_4(\widetilde{L}, M_c) = \frac{18M_c}{5\widetilde{L}} + \frac{3}{5}} \tag{1402}$$

Note from eq.(1401) that, at $h = 0$, the nominal moment is negative, due to the restoring force increasing as x increases. Thus the robustness curve sprouts off the negative axis, so the design starts out more robust than the previous designs. However, the slope is lower so the cost of robustness is higher.

(e) Designer 5: $R(x) = 2(1-x) \int_0^1 L(y)\,dy$. (Note: $\int_0^1 x \times 2(1-x)\,dx = 1/3$.)

$$\begin{aligned} \mu(h) &= \max_{L\in\mathcal{U}(h)} \int_0^1 x[L(x) - R(x)]\,dx & (1403)\\ &= \max_{L\in\mathcal{U}(h)} \left[\int_0^1 xL(x)\,dx - \int_0^1 x \times 2(1-x)\,dx \int_0^1 L(y)\,dy\right] & (1404)\\ &= \max_{L\in\mathcal{U}(h)} \int_0^1 \left[\left(x - \frac{1}{3}\right) L(x)\,dx\right] & (1405) \end{aligned}$$

$x - \frac{1}{3}$ is negative for $x < \frac{1}{3}$ and positive otherwise. Thus the maximum occurs for:

$$L(x) = \begin{cases} (1-h)\widetilde{L} & \text{if } x < \frac{1}{3} \\ (1+h)\widetilde{L} & \text{else} \end{cases} \tag{1406}$$

Thus:

$$\mu(h) = (1-h)\widetilde{L} \int_0^{1/3} \left(x - \frac{1}{3}\right) \widetilde{L}\,dx + (1+h)\widetilde{L} \int_{1/3}^1 \left(x - \frac{1}{3}\right) \widetilde{L}\,dx = \cdots = \frac{\widetilde{L}}{56} + \frac{8\widetilde{L}h}{56} \tag{1407}$$

Equate this to M_c and solve for h to obtain the robustness:

$$\boxed{\widehat{h}_5(\widetilde{L}, M_c) = \frac{7M_c}{\widetilde{L}} - \frac{1}{8}} \tag{1408}$$

Note from eq.(1407) that, at $h = 0$, the nominal moment is positive, due to the restoring force decreasing as x increases. Thus the robustness curve sprouts off the positive axis, so the design starts out less robust than the previous designs. However, the slope is greater than designs 1–3, so the cost of robustness is lower. Thus there is a preference reversal between this design and designs 1–3.

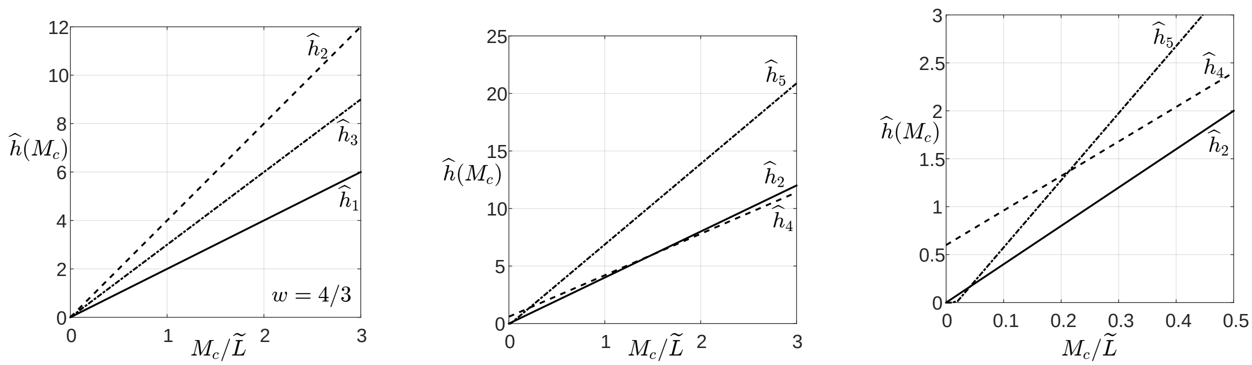

Figure 73: Robustness curves for problem 55.

Fig. 73 shows the 5 robustness curves. The left frame shows that $\widehat{h}_2 > \widehat{h}_3 > \widehat{h}_1$ for all positive values of M_c, as expected from eqs.(1387), (1393) and (1396). The middle frame shows that $\widehat{h}_4$ is most robust at low M_c, but the low cost of robustness of $\widehat{h}_5$ rapidly dominates, resulting in a preference reversal between these designs. The righthand frame is an expanded version of the middle frame illustrating that $\widehat{h}_5$ is the least robust at very low M_c but the most robust subsequently.