13. **Single hypothesis test,** p.6 (p.19). A particular property, x, (e.g. height, temperature, longevity, etc.) of a healthy population is a random variable with known mean and variance μ and σ^2. The central limit theorem asserts that, regardless of the distribution of x, the mean, $\overline{x}$, of a large random sample of size N is normally distributed with mean μ and variance σ^2/N. Given an observed value of the sample mean, $\overline{x}_{\rm obs}$, we want to test the hypothesis that the population is healthy:

$$H_0: \quad \overline{x} \sim \mathcal{N}(\mu, \sigma^2/N) \tag{17}$$

We will reject H_0 if $\overline{x}_{\rm obs}$ is an implausible value, conditioned on H_0. Specifically, we reject H_0 if, conditioned on H_0:

$$\text{Prob}\left(\left| \frac{\overline{x}-\mu}{\sigma/\sqrt{N}} \right| > \left| \frac{\overline{x}_{\rm obs}-\mu}{\sigma/\sqrt{N}} \right| ; \ H_0 \right) \le \alpha \tag{18}$$

where α is a 'level of significance'. If H_0 holds and if the sample is statistically random, then the distribution of $\frac{\overline{x}-\mu}{\sigma/\sqrt{N}}$ would be standard normal, $\mathcal{N}(0,1)$. Let $\Phi(z)$ denote the cumulative probability distribution (CPD) for $\mathcal{N}(0,1)$. The problem is that we are unsure that the sample is truly random: statistically independent measurements from the same population. Thus we are unsure that the true distribution of $\frac{\overline{x}-\mu}{\sigma/\sqrt{N}}$, call it $F(\cdot)$, is actually $\Phi(\cdot)$. We represent this uncertainty with the following info-gap model in which we introduce a simplifying assumption that the distributions are symmetric around the origin:

$$\mathcal{U}(h) = \left\{ F(z): \ F(-\infty)=0, \ F(\infty)=1, \ F(z)=1-F(-z), \ \frac{{\rm d}F}{{\rm d}z} \ge 0, \ |F(z)-\Phi(z)| \le h \right\}, \quad h \ge 0 \tag{19}$$

(a) We have an observed value of the sample mean, $\overline{x}_{\rm obs}$. Suppose that eq.(18) holds based on this observation, implying that we should reject H_0. How much can the distribution of $\frac{\overline{x}-\mu}{\sigma/\sqrt{N}}$ deviate from $\mathcal{N}(0,1)$ without changing this decision? That is, derive the robustness as a function of the rejection threshold, α.

(b) In contrast to part 13a, suppose that we have observed a sample mean, $\overline{x}_{\rm obs}$, for which eq.(18) does not hold, implying that we should accept H_0. Derive the robustness as a function of the rejection threshold, α.