Lecture 4

Statistical Inference

with

Info-Gaps in Underlying Processes

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Contents

1 *"Temperament" Measurement: Testing a Sample Mean*

1.1 *The Question*

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§ We wish to sample a population, to estimate T:
Temperament, Temperature, emotional Turbulence, etc.

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§ **What does "really has temperament T" *mean*?**

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Temperament, Temperature, emotional Turbulence, etc.

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- **Statistical variability.**
- **Variation of sub-populations.**
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§ How do we use these measurements to decide if the population "really" has temperament T?

§ What does "really has temperament T" *mean*?

§ Perhaps: $\mathrm{E}(T) = \mu$. (The mean is the meaning???)

1.2 *Statistical Formulation*

§ Random sample, definition:

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  Set of independent measurements from the same population.

- **More technically:**

  Set of independent and identically distributed (i.i.d.) random variables.

§ The sample mean is defined as:

$$\overline{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{1}$$

1.3 *First Hypothesis Test: Normal Distribution*

§ Suppose:

- **The temperament is normally distributed.**

or

-

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- **The number of measurements is large.**

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- **The number of measurements is large.**

§ Thus:

$$\overline{x} \sim \mathcal{N}(\mu, \sigma^2/N) \tag{2}$$

where:

μ = **true mean temperament.**

σ^2 = **variance of the temperament measurements.**

N = **sample size.**

§

§ **Suppose:**

- **The temperament is normally distributed.**

or

- **The number of measurements is large.**

§ **Thus:**

$$\overline{x} \sim \mathcal{N}(\mu, \sigma^2/N) \quad (3)$$

where:

μ = **true mean temperament.**

σ^2 = **variance of the temperament measurements.**

N = **sample size.**

§ **Also, assume that we know the value of** σ^2**.**

§ How do we decide if the true temperament is T?

§

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§ **We use an hypothesis test.**

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§ **The null hypothesis:**

$$H_0: \qquad \mu = T \tag{4}$$

T = **hypothesized temperament: a known value.**

μ = **true temperament: an unknown value.**

§

§ **How do we decide if the true temperament is T?**

§ **We use an hypothesis test.**

§ **The null hypothesis:**

$$H_0: \qquad \mu = T \tag{5}$$

T = **hypothesized temperament: a known value.**

μ = **true temperament: an unknown value.**

§ **The alternative hypothesis:**

$$H_1: \qquad \mu \neq T \tag{6}$$

This is a two-tailed test.

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§ How do we decide if the true temperament is T?

§ We use an hypothesis test.

§ The null hypothesis:

$$H_0 : \qquad \mu = T \tag{7}$$

T = hypothesized temperament: a known value.

μ = true temperament: an unknown value.

§ The alternative hypothesis:

$$H_1 : \qquad \mu \neq T \tag{8}$$

This is a two-tailed test.

§ For a one-tailed the alternative hypothesis would be:

$$H_1 : \qquad \mu > T \tag{9}$$

or:

$$H_1 : \qquad \mu < T \tag{10}$$

§ Level of significance, α:

- **Probability of obtaining a result at least as extreme as the observed result, conditioned upon H_0.**
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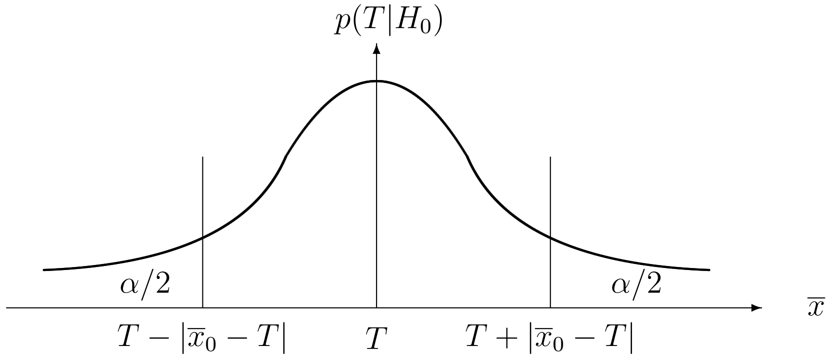

Figure 1: Sketch of probability density, illustrating the level of significance in eq.(11).

§ For the two-tailed test (fig. 1):

$$\alpha = \textbf{Prob}\left(|\overline{x}-T| \geq |\overline{x}_o - T| \;\Big|\; H_0\right) \tag{11}$$

$\overline{x}$ **= the random variable "sample mean".**

$\overline{x}_o$ **= the observed value of the random variable $\overline{x}$.**

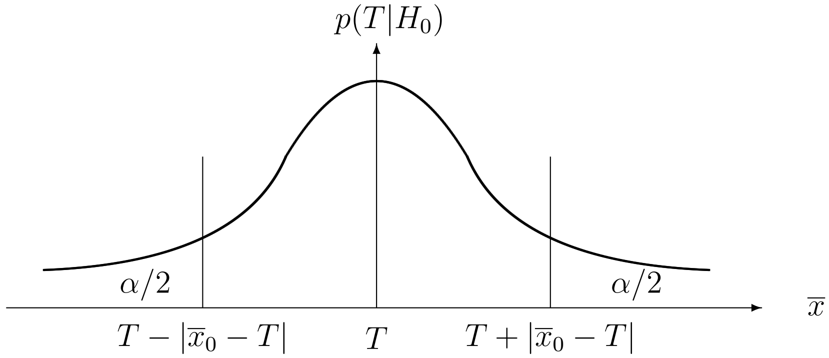

Figure 2: Sketch of probability density, illustrating the level of significance in eq.(11).

§ Interpreting the level of significance:

If α is small: reject H_0.

If α is large: accept H_0.

§ How to evaluate the level of significance?

§

§ How to evaluate the level of significance?

§ Conditioned upon H_0, we can assert

(recall we assume normal measurements or large sample size):

$$\overline{x} \sim \mathcal{N}(T, \sigma^2/N) \tag{12}$$

$\overline{x}$ **can be standardized as:**

$$z = \frac{\overline{x} - T}{\sigma/\sqrt{N}} \sim \mathcal{N}(0, 1) \tag{13}$$

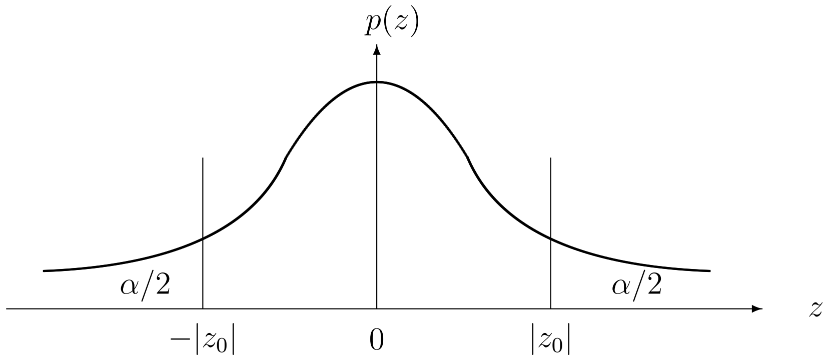

Figure 3: Sketch of probability density illustrating level of significance in eq.(16).

§ The level of significance can be expressed as (fig. 3):

$$\alpha = \textbf{Prob}\left(|\overline{x} - T| \geq |\overline{x}_o - T| \,\Big|\, H_0\right) \tag{14}$$

$$= \textbf{Prob}\left(\frac{|\overline{x} - T|}{\sigma/\sqrt{N}} \geq \frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}} \,\Big|\, H_0\right) \tag{15}$$

$$= 2\left[1 - \Phi\left(\frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}}\right)\right] \tag{16}$$

$\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution.

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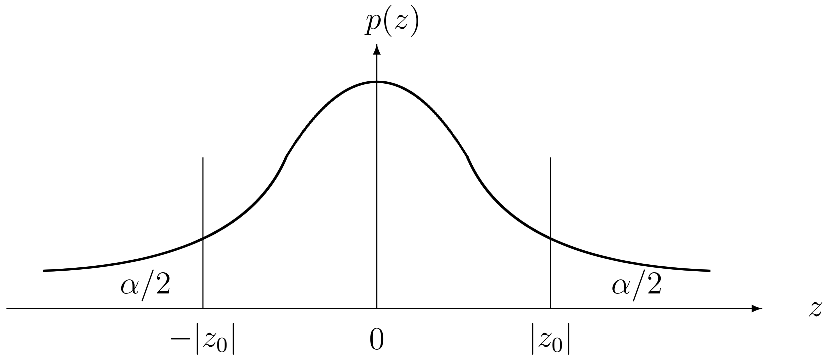

Figure 4: Sketch of probability density illustrating level of significance in eq.(19).

§ The level of significance can be expressed as (fig. 4):

$$\alpha = \mathbf{Prob}\left(|\overline{x} - T| \geq |\overline{x}_o - T| \;\middle|\; H_0\right) \tag{17}$$

$$= \mathbf{Prob}\left(\frac{|\overline{x} - T|}{\sigma/\sqrt{N}} \geq \frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}} \;\middle|\; H_0\right) \tag{18}$$

$$= 2\left[1 - \Phi\left(\frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}}\right)\right] \tag{19}$$

$\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution.

- **Thus:**

$$z_o = \frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}} \quad \Longrightarrow \quad \alpha = 2[1 - \Phi(z_o)] \tag{20}$$

1.4 *Info-Gap Uncertain Distribution Function*

§ The problem.

- **The level of significance in eq.(16):**

$$\alpha = \left[1 - \Phi \left(\frac{|\overline{x}_o - T|}{\sigma / \sqrt{N}} \right) \right] \tag{21}$$

-

§ The problem.

- **The level of significance in eq.(16):**

$$\alpha = \left[1 - \Phi \left(\frac{|\overline{x}_o - T|}{\sigma / \sqrt{N}} \right) \right] \tag{22}$$

- **Assumes the observations are all normal or i.i.d.**
-

§ The problem.

- **The level of significance in eq.(16):**

$$\alpha = \left[1 - \Phi\left(\frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}}\right)\right] \tag{23}$$

- **Assumes the observations are all normal or i.i.d.**
- **These assumptions may be wrong due to:**
  - **Population heterogeneity.**
  -

§ The problem.

- **The level of significance in eq.(16):**

$$\alpha = \left[1 - \Phi\left(\frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}}\right)\right] \qquad (24)$$

- **Assumes the observations are all normal or i.i.d.**
- **These assumptions may be wrong due to:**
  - **Population heterogeneity.**
  - **Population change during the sampling.**
  -

§ The problem.

- **The level of significance in eq.(16):**

$$\alpha = \left[1 - \Phi \left(\frac{|\overline{x}_o - T|}{\sigma / \sqrt{N}} \right) \right] \tag{25}$$

- **Assumes the observations are all normal or i.i.d.**
- **These assumptions may be wrong due to:**
  - **Population heterogeneity.**
  - **Population change during the sampling.**
  - **Non-random sampling procedures.**
-

§ The problem.

- **The level of significance in eq.(16):**

$$\alpha = \left[1 - \Phi\left(\frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}} \right) \right] \tag{26}$$

- **Assumes the observations are all normal or i.i.d.**
- **These assumptions may be wrong due to:**
  - **Population heterogeneity.**
  - **Population change during the sampling.**
  - **Non-random sampling procedures.**
- **Hence the cdf, $\Phi(z)$, may be wrong.**

§

§ The problem.

- **The level of significance in eq.(16):**

$$\alpha = \left[1 - \Phi\left(\frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}}\right)\right] \qquad (27)$$

- **Assumes the observations are all normal or i.i.d.**
- **These assumptions may be wrong due to:**
  - **Population heterogeneity.**
  - **Population change during the sampling.**
  - **Non-random sampling procedures.**
- **Hence the cdf, $\Phi(z)$, may be wrong.**

§ Uncertainty:

- **Extent of error of $\Phi(z)$ unknown.**
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§ The problem.

- **The level of significance in eq.(16):**

$$\alpha = \left[1 - \Phi\left(\frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}}\right)\right] \quad (28)$$

- **Assumes the observations are all normal or i.i.d.**
- **These assumptions may be wrong due to:**
  - **Population heterogeneity.**
  - **Population change during the sampling.**
  - **Non-random sampling procedures.**
- **Hence the cdf, $\Phi(z)$, may be wrong.**

§ Uncertainty:

- **Extent of error of $\Phi(z)$ unknown.**
- **No probabilistic information about this uncertainty.**
- **This is an info-gap.**

§ Info-gap model of uncertainty.

- $F(z)$ = **unknown correct cdf.**
-

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- Info-gap model of non-probabilistic uncertainty:

$$\mathcal{U}(h) = \left\{ F(z): \quad F(-\infty)=0,\ F(\infty)=1,\ \frac{\partial F(z)}{\partial z} \geq 0, \right. \\ \left. \left| \frac{F(z)-\Phi(z)}{\Phi(z)} \right| \leq h \right\}, \quad h \geq 0 \tag{29}$$

- First 3 conditions specify properties of any cdf.
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- **First 3 conditions specify properties of any cdf.**
- **4th condition states:**
  fractional error of $\Phi(x)$ is no greater than h.
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- First 3 conditions specify properties of any cdf.
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- However, h is unknown: $h \geq 0$:
  - Unbounded horizon of uncertainty.
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- First 3 conditions specify properties of any cdf.
- 4th condition states:
  fractional error of $\Phi(x)$ is no greater than h.
- However, h is unknown: $h \geq 0$:
  - Unbounded horizon of uncertainty.
  - No known worst case.

- **The info-gap model of uncertainty,** $\mathcal{U}(h), h \geq 0$ **is:**
  - **Unbounded family of nested sets of possible cdf's.**
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- **Two properties of all info-gap models of uncertainty:**

$$\mathcal{U}(0) = \{\Phi(z)\}, \qquad \textbf{(Contraction)} \tag{33}$$

$$h < h' \implies \mathcal{U}(h) \subseteq \mathcal{U}(h'), \qquad \textbf{(nesting)} \tag{34}$$

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- **The info-gap model of uncertainty, $\mathcal{U}(h), h \geq 0$ is:**
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- **Two properties of all info-gap models of uncertainty:**

$$\mathcal{U}(0) = \{\Phi(z)\}, \qquad (\textbf{Contraction}) \tag{35}$$

$$h < h' \implies \mathcal{U}(h) \subseteq \mathcal{U}(h'), \qquad (\textbf{nesting}) \tag{36}$$

§ **Known estimated and unknown true**

levels of significance are:

$$\alpha(\Phi) = 2\left[1 - \Phi\left(\frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}}\right)\right] \tag{37}$$

$$\alpha(F) = 2\left[1 - F\left(\frac{|\overline{x}_o - T|}{\sigma/\sqrt{N}}\right)\right] \tag{38}$$

§ A common requirement:

- **We are concerned that the estimate, $\alpha(\Phi)$, is overly optimistic: an under-estimate of the unknown true level of significance, $\alpha(F)$.**
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- **We are concerned that the estimate, $\alpha(\Phi)$, is overly optimistic: an under-estimate of the unknown true level of significance, $\alpha(F)$.**
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$$\alpha(F) - \alpha(\Phi) \le \varepsilon \tag{39}$$

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§ Robustness:

- **We don't know how much $\Phi(z)$ errs.**
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§ Robustness:

- **We don't know how much $\Phi(z)$ errs.**
- **We do know how much error we can tolerate: eq.(41)**
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§ A common requirement:

- **We are concerned that the estimate, $\alpha(\Phi)$, is overly optimistic: an under-estimate of the unknown true level of significance, $\alpha(F)$.**
- **We can accept $\alpha(\Phi)$ if it errs no more than ε:**

$$\alpha(F) - \alpha(\Phi) \le \varepsilon \qquad (42)$$

§ Robustness:

- **We don't know how much $\Phi(z)$ errs.**
- **We do know how much error we can tolerate: eq.(42)**
- **The robustness to uncertainty in the cdf is the max horizon of uncertainty, h, up to which the error is tolerable . . .**

§ A common requirement:

- We are concerned that the estimate, $\alpha(\Phi)$, is overly optimistic: an under-estimate of the unknown true level of significance, $\alpha(F)$.
- We can accept $\alpha(\Phi)$ if it errs no more than ε:

$$\alpha(F) - \alpha(\Phi) \leq \varepsilon \tag{43}$$

§ Robustness:

- We don't know how much $\Phi(z)$ errs.
- We do know how much error we can tolerate: eq.(43)
- The robustness to uncertainty in the cdf is the max horizon of uncertainty, h, up to which the error is tolerable:

$$\begin{aligned}\widehat{h}(\varepsilon) &= \text{maximum tolerable uncertainty in } F(z). \\ &= \text{maximum } h \text{ such that eq.(43) is satisfied.} \\ &= \max\left\{h : \max_{F\in\mathcal{U}(h)} (\alpha(F) - \alpha(\Phi)) \leq \varepsilon\right\} \end{aligned} \tag{44}$$

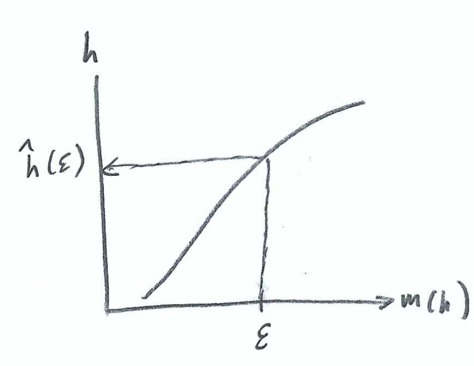

Figure 5: Schematic plot of h vs. $m(h)$.

§ $m(h)$ = inner maximum in definition of robustness:

$$m(h) = \max_{F \in \mathcal{U}(h)} \left(\alpha(F) - \alpha(\Phi) \right) \qquad (45)$$

This function has two properties (fig. 5):

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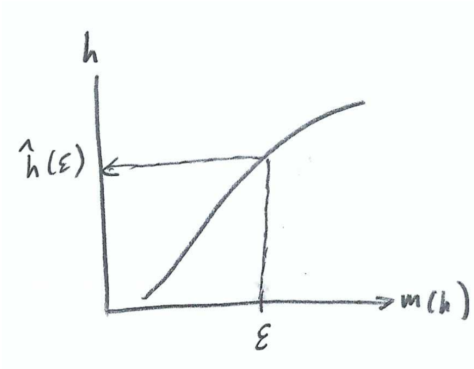

Figure 6: Schematic plot of h vs. $m(h)$.

§ $m(h)$ = inner maximum in definition of robustness:

$$m(h) = \max_{F \in \mathcal{U}(h)} \left(\alpha(F) - \alpha(\Phi)\right) \tag{46}$$

This function has two properties (fig. 6):

- **$m(h)$ increases as h increases.**
-

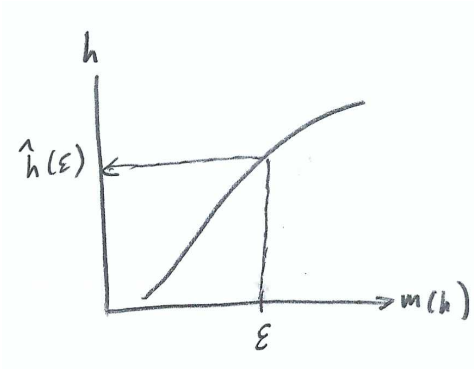

Figure 7: Schematic plot of h vs. $m(h)$.

§ $m(h)$ = **inner maximum in definition of robustness:**

$$m(h) = \max_{F \in \mathcal{U}(h)} (\alpha(F) - \alpha(\Phi)) \qquad (47)$$

This function has two properties (fig. 7):

- $m(h)$ **increases as** h **increases.**
- $m(h)$ **is the inverse function of** $\widehat{h}(\varepsilon)$**:**

  a plot of h **vs** $m(h)$

  is the same as

  a plot of $\widehat{h}(\varepsilon)$ **vs** ε**.**

§ Two properties of the robustness:

$$\widehat{h}(\varepsilon) = \frac{\varepsilon}{2\Phi(Q)} \tag{48}$$

§ Zeroing in eq.(48):

No robustness ($\widehat{h} = 0$) when requiring no error ($\varepsilon = 0$).

§ Trade off in eq.(48):

Robustness increases as greater error is allowed.

1.5 *Second Hypothesis Test: t Distribution*

§ We tested H_0 (p.25) while assuming that σ^2 is known.

- **What do we do if σ^2 is unknown?**
-

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- **What do we do if σ^2 is unknown?**
- **We still assume that the sample is normal.**

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§ **We tested H_0 while assuming that σ^2 is known.**

- **What do we do if σ^2 is unknown?**
- **We still assume that the sample is normal.**

§ **Two cases:**

- N **large.**
- N **small.**

§ Case 1: N large.

- **Sample variance:**

$$s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \overline{x})^2 \tag{49}$$

-

§ Case 1: N large.

- **Sample variance:**

$$s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \overline{x})^2 \tag{50}$$

- **If $N \geq 25$ (rough number), then s^2 is a good estimate of the true population variance, σ^2.**

-

§ Case 1: N large.

- **Sample variance:**

$$s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \overline{x})^2 \qquad (51)$$

- **If $N \geq 25$ (rough number), then s^2 is a good estimate of the true population variance, σ^2.**

- **We can now assume, conditioned on H_0, that:**

$$t = \frac{\overline{x} - T}{s/\sqrt{N}} \sim \mathcal{N}(0, 1) \qquad (52)$$

This assumption would be precise if $s^2 = \sigma^2$.

Now we proceed with the test as before.

§ Case 2: N small.

- **If $N < 25$ (rough number), then s^2 is not a good estimate of σ^2.**
-

§ Case 2: N small.

- **If $N < 25$ (rough number), then s^2 is not a good estimate of σ^2.**
- **The statistic:**

$$t = \frac{\overline{x} - T}{s/\sqrt{N}} \tag{53}$$

is broader than $\mathcal{N}(0,1)$ because $\overline{x}$ and s both display variation.

-

§ Case 2: N small.

- **If $N < 25$ (rough number), then s^2 is not a good estimate of σ^2.**
- **The statistic:**

$$t = \frac{\overline{x} - T}{s/\sqrt{N}} \tag{54}$$

is broader than $\mathcal{N}(0, 1)$ because $\overline{x}$ and s both display variation.

- **This is a t statistic with $N - 1$ degrees of freedom.**
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§ Case 2: N small.

- **If $N < 25$ (rough number), then s^2 is not a good estimate of σ^2.**
- **The statistic:**

$$t = \frac{\overline{x} - T}{s/\sqrt{N}} \tag{55}$$

is broader than $\mathcal{N}(0,1)$ because $\overline{x}$ and s both display variation.

- **This is a t statistic with $N-1$ degrees of freedom.**
- **Recall we assumed the sample is normal.**

§ **We repeat the numerical example, not knowing σ^2.**

- **Measurements: 1.3, 1.2, 1.4, 1.3, 1.1**
-

§ We repeat the numerical example, not knowing σ^2.

- **Measurements: 1.3, 1.2, 1.4, 1.3, 1.1**
- **Hypothesized temperament:** $T = 1.36$**.**
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§ **We repeat the numerical example, not knowing σ^2.**

- **Measurements: 1.3, 1.2, 1.4, 1.3, 1.1**
- **Hypothesized temperament:** $T = 1.36$.
- $N = 5$ **and** $\overline{x}_o = 1.26$.
-

§ We repeat the numerical example, not knowing σ^2.

- **Measurements: 1.3, 1.2, 1.4, 1.3, 1.1**
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- **Sample variance:** $s^2 = 0.013 \quad \Longrightarrow \quad s = 0.1140$.
- **The observed t statistic is:**

$$t_o = \frac{\overline{x} - T}{s/\sqrt{N}} = \frac{1.26 - 1.36}{0.1140/\sqrt{5}} = -1.961 \tag{56}$$

- **The dofs:** $5 - 1 = 4$.

- **Table of the t distribution:**

$\alpha =$	**0.1**	**0.05**	**0.025**	**0.01**
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With 4 dofs:

- **The probability of t_4 exceeding 1.533 is 0.1.**
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$$\alpha = \textbf{Prob}\left(|t| \geq |t_o| \, \middle| \, H_0 \right) \approx 2 \times 0.07 = 0.14 \qquad (57)$$

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- **This is not small, so we cannot reject H_0.**
- **The probability of falsely rejecting H_0 is 0.14.**

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Any Questions?