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ORIGINAL ARTICLE

Evidence and uncertainty: An info-gap analysis of uncertainty-augmenting evidence

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Decisions in many disciplines are based on understanding and evidence. More evidence is better than less when it enhances the decision-maker's understanding. This is achieved by reducing uncertainty confronting the decision-maker and reducing the potential for misunderstanding and failure. However, some evidence may actually augment uncertainty by revealing prior error or ignorance. True evidence that augments uncertainty is important because it identifies inadequacies of current understanding and may suggest directions for rectifying this. True evidence that reduces uncertainty may simply reconfirm or strengthen prior understanding. Uncertainty-augmenting evidence, when it is true, can support the expansion of one's previously incomplete understanding. A dilemma arises because both reduction and enhancement of uncertainty can be beneficial, and both are not simultaneously possible on the same issue. That is, uncertainty can be either pernicious or propitious. Info-gap theory provides a response. The info-gap robustness function enables protection against pernicious uncertainty by inhibiting failure. The info-gap opportuneness function enables exploitation of propitious uncertainty by facilitating wonderful windfall outcomes. The dilemma of uncertainty-augmenting evidence is that robustness and opportuneness are in conflict; a decision that enhances one, worsens the other. This antagonism between robustness and opportuneness—between protecting against pernicious uncertainty and exploiting propitious uncertainty—is characterized in a generic proposition and corollary. These results are illustrated in an example of allocation of limited resources.

KEYWORDS
decision making, evidence, opportuneness, robustness, uncertainty

1 | THE PROBLEM

Decisions in many disciplines are based on understanding and evidence. For example, evidence-based medicine attempts to ground medical decisions on solid scientific evidence. Many other fields advocate evidence-based practice, including architecture, education, engineering, law, management, and public policy, among others. In engineering, for instance, designs are based on science-based models of the processes involved, and evidence—data—relevant to the problem at hand.

Evidence is intended to enhance decision-makers' understanding. This is often achieved by reducing uncertainty confronting the decision-maker. However, some evidence may actually augment uncertainty. Febrile episodes are common in patients with acute leukemia, usually caused by infection (Bodey et al., 1978). After detecting fever, the diagnosis proceeds by characterizing the infection, until evidence shows absence of infection; something else is causing the fever. The engineering designer thought that stiffening the structure would enhance stability, until evidence revealed harmonic oscillations in the environment of the structure;

stiffening may not shift the structure's natural frequencies of vibration away from the environmental frequencies. Evidence that augments uncertainty is important because it identifies inadequacies of current understanding and may suggest directions for rectifying this.

Curiosity is a desire for new information. This curiosity may occur in two types. Interest-type curiosity "is motivated by stimulating positive feelings of enjoyment and excitement associated with discovery and understanding. Deprivation-type curiosity… is motivated by reducing negative feelings of frustration and tension associated with a specific source of uncertainty, ignorance, and misunderstanding" (Whitecross & Smithson, 2023, p. 2).

Interest-type curiosity seeks pleasures from new and unfamiliar knowledge. Deprivation-type curiosity seeks protection against adverse feelings that arise from uncertainty and misunderstanding. Interest-type curiosity seeks to benefit from the unknown, while deprivation-type curiosity seeks to limit the adverse impact of ignorance.

Interest-type curiosity could motivate the individual to enhance the opportuneness of uncertainty, while deprivation-type curiosity could motivate the individual to enhance the robustness against uncertainty. Interest-type curiosity may motivate the engineer to apply very large loads in attempting to discover conceptual flaws in the design model, or motivate the central bank to dramatically lower interest rates to explore market responsiveness, or motivate the physician to reduce therapy to test the patient's resilience. These actions motivated by interest-type curiosity are all seeking informative results. Deprivation-type curiosity would motivate the agent—engineer, economist, physician—to the other extreme in seeking to assure stable or adequate outcomes. Opportuneness from uncertainty and robustness against uncertainty underlie our exploration of decision making when facing the dilemma of evidence and uncertainty.

Thekdi and Aven (2023) discuss a range of liabilities facing risk analysts, stressing that "[t]he practice of risk analysis is critical for directing investments toward prevention and mitigation for uncertain risk events" (p. 1212). Part of the risk analyst's task is addressing "[w]eak knowledge of the system" (p. 1215). Thus, part of the risk analyst's task is directing investments to obtaining new evidence that will improve knowledge of the system. Decisions that enhance robustness against failure are important, but decisions that enable wonderful outcomes are also relevant in situations of deep uncertainty about significant outcomes. We will see an inherent conflict in these two types of decisions, which must be balanced by the risk analyst in managing uncertainty.

Uncertainty can be either pernicious or propitious. Wu and Trump (2023) write that "[e]very decision carries some prospect of possible gains or harms, with our efforts intended to seek the former and mitigate the latter…. Through a combination of choice and chance, we may achieve far better outcomes when selecting a promising but more uncertain choice than what is conventional. Of course, bad luck can lead to losses" (p. 871). Our study of robustness and opportuneness, as formulated in info-gap theory, will expose an inherent conflict between these two possibilities when managing uncertainty in support of decision.

Hagiwara et al. (2023) discuss a value-of-information methodology "to compare and rank different tests that could be used to characterize the toxicity of chemical substances… for consideration of the value of the public health benefits derived from risk mitigation actions that may be taken based on the test results" (p. 512). They focus on "the degree of uncertainty reduction provided by different tests for chemical toxicity" (p. 499). Public health benefits can result not only from uncertainty reduction, but also from opportunities for exploiting uncertainty for surprisingly favorable outcomes. We explore the tension between robustly reducing the impact of pernicious uncertainty and exploiting the opportunities of propitious uncertainty.

It is not surprising that evidence can augment uncertainty in situations where meanings or intentions are of primary importance. Discovery of Soviet nuclear missiles in Cuba in 1962 induced great uncertainty regarding Soviet intentions (Allison & Zelikow, 1999, pp. 33, 49). Discovering in 1994 that CIA agent Aldrich Ames had been a double agent for the KGB since 1985 raised great uncertainty about the implications of revealed secrets (Diamond, 2008).

It is perhaps more surprising that evidence can augment uncertainty when mathematical structure—rather than semantic meaning—is dominant. Such situations are prevalent in engineering design and analysis, or in quantitative modeling of physical or social systems, or in economic policy formulation based on macroeconomic models, and so forth.

It is not surprising that conflicting evidence augments uncertainty. In conflicting evidence, one part asserts "Proposition X is true" while another part asserts "Proposition X is not true." One of these assertions is not correct. However, we are considering true evidence from which such conflict is precluded.

Decision making under uncertainty has confronted humanity since ancient times, inducing a plethora of methodological responses ranging from many forms of divination to modern probability originating in the early 17th century (Hacking, 1975), fuzzy logic (Klir & Folger, 1988), Dempster–Shafer theory (Shafer, 1966), info-gap theory (Ben-Haim, 2006), and more. Each methodology presumes specific types of prior knowledge. Probability and resulting statistical decision tools presume knowledge of probability distributions, or at least knowledge of properties of the uncertain process such as independence, identical distribution, and so forth. Fuzzy logic uses membership functions rather than probability distributions, and Dempster–Shafer theory replaces probability distributions with belief functions, where each type of function reflects different ambiguous semantic contextual understanding. Info-gap theory is motivated by substantial lack of understanding of the phenomena of interest, and represents uncertainty with an unbounded family of nested sets of possible realizations of the uncertain entity. Info-gap theory usually provides less insight into the uncertainty than the other methods, but nonetheless supports systematic decision making when the other methods are inaccessible due to limited prior knowledge. This paper employs info-gap theory.

We briefly review a range of different responses to uncertainty, explaining their similarities and differences from the info-gap approach.

Taleb's concept of antifragility presents a distinctive approach to human response to uncertainty. Taleb writes: "Antifragility is beyond resilience or robustness. The resilient resists shocks and stays the same; the antifragile gets better." (Taleb, 2014, p. 3). Antifragility is the idea that one can improve in response to adverse challenges. This is different from the concepts of robustness and opportuneness developed in info-gap theory and employed in this paper. Robustness, here, is the idea of confidently achieving essential goals despite uncertain challenges. Opportuneness is the idea of exploiting uncertainty to obtain better than anticipated outcomes. Info-gap robustness responds to pernicious uncertainty while info-gap opportuneness responds to propitious uncertainty. Antifragility is somewhat a combination, leading to positive outcomes (getting better) in response to adverse shocks, but differs from both info-gap concepts of robustness and opportuneness. Furthermore, info-gap theory provides a systematic quantitative methodology for modeling the unbounded unknown, and then prioritizing decisions in two ways: according to robustness against that uncertainty, or according to opportunities inherent in that uncertainty.

Hall and Solomatine (2008) present a broad framework for analyzing uncertainty in the study of flood risks. Their analysis "encompasses all of the natural, human and technological processes that may influence flood risk." Referring to Knight (1921), they distinguish between decision making under certainty, decision making under risk for which probability models are known, and decision making under uncertainty in which probabilities are unknown which Knight called "true uncertainty." They discuss a range of methods, including scenario analysis, sensitivity analysis, fuzzy set theory (Zadeh, 1965), evidence theory (Shafer, 1976), the theory of imprecise probabilities (Walley, 1991) and info-gap decision theory. In each case, the method is tailored to specific types of prior knowledge about the system and its uncertainties. Info-gap theory is usually the epistemically minimalistic of these approaches because it presumes no knowledge of a scalar function of probability, membership, or belief in describing one's ignorance or uncertainty. An info-gap model of uncertainty is an unbounded family of nested sets of possible realizations of the uncertain entity. Similarities exist, for instance, with Walley's imprecise probabilities that are described by sets of probability distributions.

Similarly, Li et al. (2013) compare a range of theories for decision making under uncertainty, including probability, fuzzy logic, and info-gap theory, explaining that "[d]ifferent kinds of uncertainty call for different handling methods." They explain that info-gap theory addresses "severe uncertainty… [that] is usually immeasurable or uncalculated with probability distributions and is… an incomplete understanding of the system being managed" (p. 2467). Info-gap theory "applies to the situations of limited information, especially when there is not enough data for other uncertainty handling techniques such as probability theory" (p. 2468).

Hall et al. (2012) compare and contrast Robust Decision Making as developed at RAND Corporation, and info-gap decision theory. Their analysis is both generic and applied to formulating policies for responding to climate change. They identify similarities of these methods, including nonprobabilistic representation of uncertainty with "sets of multiple plausible representations of the future, rather than a unique probability density function." Both methods "incorporate the concept of robust satisficing," that is, achieving specified required outcomes while protecting against adverse uncertainty. Both methods use "quantified system models" relating decisions to consequences. Both methods support decision making with "trade-off curves comparing alternative strategies rather than provide any definitive, unique ordering of options" though the axes differ in the two methods. Both methods "make broadly similar recommendations [regarding greenhouse gas management] that nonetheless differ in their particulars." However, the methods treat gains and losses differently, they take "different approaches to imprecise information," and perform analysis in different orders (pp. 13–14).

Roach et al. (2016) compare info-gap theory and robust optimization (RO) as developed by Ben-Tal et al. (2009). In their water resource example, info-gap theory analyses the robustness to uncertainty of a "set of prespecified strategies" while RO uses optimization algorithms to automatically generate and evaluate solutions. In this formulation, they conclude that RO produces "lower cost strategies" while info-gap theory "produced the more expensive Pareto strategies due to its more selective and stringent robustness analysis." (p. 1) They note, however, that they do not explore optimization of the info-gap robustness analysis (p. 11).

Rezaei et al. (2019) compare "stochastic programming, fuzzy optimization, interval optimization, and robust optimization for managing electrical energy systems. Each of these approaches has their advantages and drawbacks," requiring probability densities, membership functions, or specified uncertainty sets (p. 12). In contrast, they explain that info-gap theory requires less constrained or detailed prior knowledge of uncertainty.

Chebila (2023) studies design of "safety instrumented systems" in which "robustness assessment… [is] the heart of the framework" (p. 873). Chebila stresses that robustness analyses emphasize either regret or satisficing, identifying info-gap theory with the latter. (One notes, however, the regret can be satisficed rather than minimized with the info-gap approach.)

This brings us to the central questions of this paper. How to characterize, model, and manage the dilemma that arises from uncertainty that may be either pernicious or propitious, in support of decision making? This is illustrated in the example in Section 2. Proposition 1 and Corollary 1 in Section 3 generalize this example.

2 | EXAMPLE: ALLOCATION OF RESOURCES

Consider allocation of limited resources among N categories. The categories may be different endangered species and the resources are hectares of added nature reserve in each species' habitat. Or the categories may be distinct financial assets and the resources are financial investments. The categories may be locations on a mechanical structure and the resources are forces applied at each location. In any case, S is the vector of N returns or benefits for each unit of resource allocated to each category: number of surviving individuals per hectare allocated for each species, or dollar return per dollar invested in each asset, or displacement of the structure per unit load at each location. Uncertainty often accompanies the allocation in these and other applications. This uncertainty may be pernicious or propitious, and this paper explores the methodological implications of these distinct facets of uncertainty and how to make decisions in managing them.

Section 2.1 formulates the info-gap model for uncertainty in return vector S. Section 2.2 formulates the generic robustness and opportuneness functions—collectively referred to as immunity functions—and Section 2.3 presents explicit expressions for these immunity functions for a specific info-gap model of uncertainty. Section 2.4 discusses the antagonism between the robustness and opportuneness functions and Section 2.5 explores the phenomenon of preference reversal in the context of our example. Finally, Section 2.6 touches on the implications of the analysis for choosing evidence to acquire.

2.1 | Info-gap uncertainty

The vector S is uncertain and the best available estimate is $\tilde{S}$. The uncertainty in S is represented by an info-gap model, $\mathcal{U}(h)$, for $h \geq 0$, which is an unbounded family of nested sets of S vectors (Ben-Haim, 2006). The sets become more inclusive as h increases, so h is called the horizon of uncertainty.

There are many specific forms of info-gap models, each representing different (though limited) prior knowledge about the uncertain entity. For instance, consider allocation between two categories, where prior knowledge provides estimated benefits and indicates a tendency for positive association between the benefits from allocations to these two categories. The positive association may be expressed by historical covariances whose present or future values are uncertain. Or, the covariances may be from related but different populations or circumstances. In any case, even knowledge of mean and covariance is insufficient to specify a probability distribution. That is, we consider information that is insufficient to support the choice of a specific probability distribution. We will employ an info-gap model to represent uncertainty in the benefits.

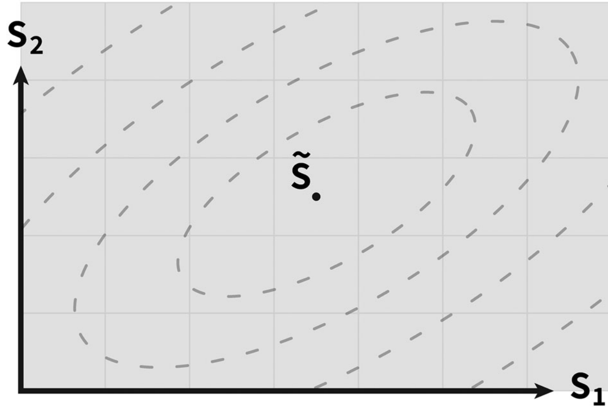

FIGURE 1 Schematic uncertainty sets of a two-dimensional ellipsoidal info-gap model of uncertainty.

The estimated vector of benefits is $\tilde{S} = (\tilde{S}_1,\ \tilde{S}_2)$, but the actual benefit vector, $S = (S_1,\ S_2)$, is unknown. Figure 1 is a schematic illustration of an ellipsoidal info-gap model for uncertainty in S. The solid dot in the center is the estimated benefit vector, $\tilde{S}$. The sequence of increasingly larger ellipses display discrete sets in the continuous family of sets representing increasingly uncertain deviation of S from $\tilde{S}$. Greater uncertainty is entailed in larger ellipses. There is no known worst case—maximum deviation between S and $\tilde{S}$—so the family of uncertainty sets grows continuously and without bound. This info-gap model for uncertainty is the unbounded family of nested ellipsoidal sets of possible vectors S.

Return to the generic case where S is the vector of N returns or benefits for each unit of resource allocated to each of N categories. An info-gap model (not necessarily ellipsoidal) for uncertainty in S is a family of sets of possible S vectors, $\mathcal{U}(h)$ for $h \geq 0$, that obeys two axioms. The first is the nesting axiom:

$$h < h' \quad \Longrightarrow \quad \mathcal{U}(h) \subseteq \mathcal{U}(h'). \tag{1}$$

This means that the range of S vectors in the uncertainty sets $\mathcal{U}(h)$ increases as h increases.

The second is the contraction axiom that asserts that, in the absence of uncertainty (namely, when $h = 0$), the uncertainty set contains only the estimated vector, $\tilde{S}$. That is:

$$\mathcal{U}(0) = \left\{\tilde{S}\right\}. \tag{2}$$

$\tilde{S}$ is called the center point of the info-gap model, and it is indeed the center point in Figure 1.

The info-gap model is unbounded: the sets $\mathcal{U}(h)$ become increasingly inclusive as the horizon of uncertainty, h, increases. The sets cover increasingly larger domains of the space as h increases, asymptotically covering the entire space as h approaches infinity. Thus, for any realization of the uncertain vector, S, there is a value of h for which S is included in the sets at all horizons of uncertainty at and beyond this value of h.

2.2 | Formulation of the robustness and opportuneness functions

The decision, D, is a vector whose nth element is the resource allocation to category n, and the consequence is the overall benefit, assessed as the sum of the benefits from each allocation:

$$C(S,D) = D^T S. \tag{3}$$

We require that the benefit be no less than an essential value, C_e:

$$C(S,D) \geq C_e. \tag{4}$$

The robustness of decision D is defined as the greatest horizon of uncertainty at which the benefit is guaranteed to achieve no less than an essential value, C_e:

$$\hat{h}(C_e, D) = \max\left\{h:\ \left(\min_{S\in\mathcal{U}(h)} C(S,D)\right) \geq C_e\right\}. \tag{5}$$

A large value of the benefit function, $C(S,D)$, is a desirable response to inputs D. A small value of $C(S,D)$ is undesirable. However, a surprisingly small value of $C(S,D)$ is informative because it reveals a lacuna in our understanding. Our knowledge and understanding of the uncertain response vector, S, are enhanced by a decision, D, at which $C(S,D)$ is low, no greater than a surprisingly small and informative value C_i:

$$C(S,D) \leq C_i. \tag{6}$$

The opportuneness is the lowest horizon of uncertainty at which $C(S,D)$ may be less than C_i:

$$\hat{\beta}'(C_i, D) = \min\left\{h:\ \left(\min_{S\in\mathcal{U}(h)} C(S,D)\right) \leq C_i\right\}. \tag{7}$$

Let D_o be a decision vector for which the opportuneness function, $\hat{\beta}'(C_i,D)$, takes a desirably small value. It would be informative to allocate resources according to D_o, perhaps on a test case, and to observe the responses of the various categories. One expects substantial differences between the observed outcomes in the N categories, $S_1D_1, , \dots, S_ND_N$, and the anticipated outcomes, $\tilde{S}_1D_1, , \dots, \tilde{S}_ND_N$. These differences could reveal insight into the limitation of the estimated response vector $\tilde{S}$.

The inner minima in Equations (5) and (7) are the same, though their meanings are different. The inner minimum in Equation (5) is interpreted as the smallest—least desirable—benefit from allocation D. Thus, $\hat{h}(C_e,D)$ is the greatest horizon of uncertainty at which essential benefit no less than C_e is guaranteed. The inner minimum in Equation (7) is interpreted as the smallest—most informative—benefit at horizon of uncertainty h; minimal benefit is maximally informative because it reveals the potential limitation of allocation D. Thus, $\hat{\beta}'(C_i,D)$ is the lowest horizon of uncertainty at which the informativeness of D may be great as expressed by C_i (which is a small and hence informative value).

The opportuneness function that is complementary to the robustness in Equation (5), in the usual $(\hat{h}, \hat{\beta})$ pair in info-gap theory (Ben-Haim, 2006, eqs. 3.15 and 3.17), is the lowest horizon of uncertainty at which allocation D is able to achieve a wonderfully large benefit, C_w:

$$\hat{\beta}(C_w, D) = \min\left\{h:\ \left(\max_{S\in\mathcal{U}(h)} C(S,D)\right) \geq C_w\right\}. \tag{8}$$

The robustness function that is complementary to the opportuneness in Equation (7), in the usual $(\hat{h}, \hat{\beta})$ pair, is the greatest horizon of uncertainty at which allocation D is guaranteed to achieve an informativeness (modest benefit) no worse (no greater) than C_e:

$$\hat{h}'(C_e, D) = \max\left\{h:\ \left(\max_{S\in\mathcal{U}(h)} C(S,D)\right) \leq C_e\right\}. \tag{9}$$

Our analysis is based on $\hat{h}(C_e,D)$ in Equation (5) and $\hat{\beta}'(C_i,D)$ in Equation (7).

2.3 | Robustness and opportuneness functions with an ellipsoidal info-gap model

The ellipsoid-bound info-gap model, illustrated schematically in Figure 1 for the two-dimensional case, is commonly used to describe uncertainty in vectors, such as S, including information about uncertain positive associations among the elements of the vector:

$$\mathcal{U}(h) = \left\{S:\ \left(S-\tilde{S}\right)^T W \left(S-\tilde{S}\right) \leq h^2\right\},\quad h \geq 0, \tag{10}$$

where W is a known real, symmetric, positive definite matrix whose eigenvalues are $\mu_1, \dots, \mu_N$ with corresponding eigenvectors $v_1, \dots, v_N$. Each set, $\mathcal{U}(h)$, is an ellipsoid of possible realizations of S. These sets become more inclusive as the horizon of uncertainty, h, increases. The value of h is unbounded, so the range of uncertainty about the true value of S is unbounded.

Expressions for the robustness and opportuneness functions defined in Equations (5) and (7), and based on the ellipsoid-bound info-gap model in Equation (10), are readily derived:

$$\hat{h}(C_e, D) = \frac{D^T\tilde{S} - C_e}{\sqrt{D^T W^{-1} D}}, \tag{11}$$

$$\hat{\beta}'(C_i, D) = \frac{D^T\tilde{S} - C_i}{\sqrt{D^T W^{-1} D}}. \tag{12}$$

Either function is defined to equal zero if the corresponding expression is negative.

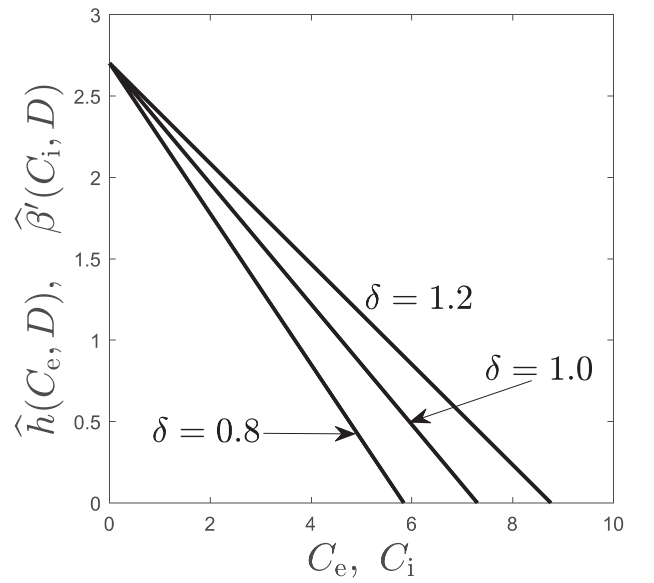

FIGURE 2 Robustness and opportuneness curves, Equations (11) and (12), illustrating the antagonism between robustness and opportuneness as the decision vector, D, changes.

2.4 | Antagonism of robustness and opportuneness

Examination of Equations (11) and (12) shows that these two immunity functions, $\widehat{h}(C_e, D)$ and $\widehat{\beta}'(C_i, D)$, are the same when each is plotted versus its performance parameter C_e or C_i. Their meanings, however, are different, and they provide different tools supporting the decision-maker as illustrated in the following numerical example.

Recall that large values of $\widehat{h}(C_e, D)$ are desirable, while small values of $\widehat{\beta}'(C_i, D)$ are desirable. Because these immunity functions are numerically the same, this means that robustness and opportuneness in this example are antagonistic: any change in the decision, D, that improves one of these functions, worsens the other. A change in D that improves the robustness (enlarges $\widehat{h}(C_e, D)$, which is desirable) also worsens the opportuneness (enlarges $\widehat{\beta}'(C_i, D)$, which is undesirable). Likewise, any change in D that improves $\widehat{\beta}'(C_i, D)$ also worsens $\widehat{h}(C_e, D)$. Proposition 1 and Corollary 1 are generic assertions of this antagonism between the robustness and opportuneness functions in Equations (5) and (7). We note that the usual pair of info-gap robustness and opportuneness functions are not necessarily antagonistic, and may be sympathetic: both functions can, in some situations, improve as a result of a change in decision (Ben-Haim, 2006).

This antagonism between robustness and opportuneness is illustrated in Figure 2 for a specific example, where the total allocation increases from the lower to the upper curve. Specifically, W is the identity matrix, there are $N = 5$ categories, and $\widetilde{S}$ and D are:

$$\widetilde{S} = (1.4,\ 1.3,\ 1.2,\ 1.1,\ 1.0), \tag{13}$$

$$D_0 = \widetilde{S}, \tag{14}$$

$$D = \delta D_0, \tag{15}$$

where $\delta = 0.8$, 1.0, and 1.2 for the bottom, middle, and top curve, respectively. We see that the robustness function increases (which is desirable) as the total allocation increases, while the opportuneness function also increases (which is undesirable).

For instance, referring to Figure 2, consider an essential benefit no less than 6. The lowest robustness curve is zero at this value, so allocation of $0.8D_0$ cannot guarantee benefit as large at 6. The robustness of the middle curve is 0.5 so allocation D_0 is guaranteed to yield a benefit at least as large as 6 provided that the horizon of uncertainty is no greater than 0.5. The robustness on the upper robustness curve is even larger, implying greater confidence in achieving a benefit no less than 6.

Now interpret the curves in Figure 2 as opportuneness curves and consider an informatively small benefit no greater than 6. With allocation as low as $0.8D_0$ (lowest curve) no uncertainty is required to enable (though not guarantee) a result no greater than 6; this allocation is quite opportune in this respect. The middle curve shows that, with allocation D_0 it is possible to achieve benefit as low as 6 only if the horizon of uncertainty is no less than 0.5, and allocation of $1.2D_0$ (upper curve) requires uncertainty as large as 0.8 to enable benefit as low as 6. By increasing the allocation, the opportunity for informatively low benefit diminishes.

In short, robustness and opportuneness are antagonistic as the allocation changes. This antagonism has an operational implication for the decision-maker who must choose the vector D of allocations among the N categories. The decision-maker must balance between the robustness requirement in Equation (4) that the overall benefit be no less than the essential value C_e, and the opportuneness aspiration in Equation (6) that the overall benefit be no greater than the small and informative value C_i.

The decision-maker may be inclined to employ the largest available budget in order to maximize the confidence in achieving an essential outcome by maximizing the robustness to uncertainty. However, surprising and informative outcomes are also valuable, and the immunity functions in Figure 2 show how to one can "buy" informative opportuneness by "selling" robustness that guarantees essential outcomes. For instance, if robustness of 0.5 is adequate (based on contextual understanding) for the essential outcome of 6, then the decision-maker may choose to allocate D_0 (rather than $1.2D_0$), thus reducing the robustness function from 0.8 to 0.5 (which is undesirable) while also reducing the opportuneness function for smaller and more informative outcomes (which is beneficial).

The assessment of how much robustness is adequate is a delicate judgment that can employ several alternative approaches: normalization of the robustness function to a dimensionless form whose value is intuitively interpretable; reasoning in analogy to similar but different situations;

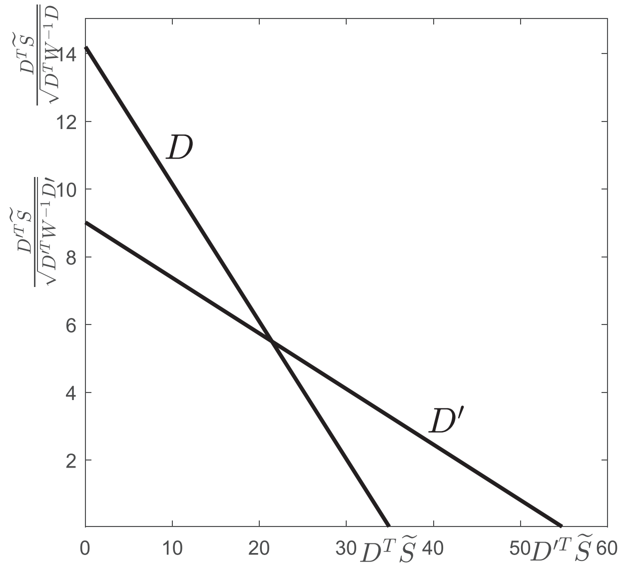

FIGURE 3 Robustness and opportuneness curves illustrating preference reversal between two allocations, D and D'.

and assessing the severity of the consequences of the outcome. These approaches are discussed extensively elsewhere (Ben-Haim, 2006, Chap. 4).

2.5 | Preference reversal

In Section 2.4, we illustrated the trade-off between robustness against uncertainty and opportuneness from uncertainty. We explained that the analyst can alter the allocation in order to "sell" some robustness and thereby "buy" some opportuneness. It is important, however, to appreciate that a change in the allocation vector D may *reduce* the robustness for some levels of essential benefit (as expressed by the requirement in Equation 4) and also *increase* the robustness for other levels of essential benefit. This is manifested in the phenomenon of preference reversal that we now illustrate. Appreciation of preference reversal is important in the analyst's choice between alternative allocations while considering the antagonism between robustness and opportuneness.

Figure 3 shows two immunity functions: The robustness, $\widehat{h}(C_e, D)$, or opportuneness, $\widehat{\beta}'(C_i, D)$, on the vertical axis, versus the required or aspired benefit, C_e or C_i, on the horizontal axis. These immunity functions are calculated for two different allocation vectors, specified below.

Considering them as robustness functions, the negative slopes of the curves express an irrevocable trade-off: as the required benefit, C_e, *increases* (becoming more demanding), the robustness for guaranteeing that benefit *decreases* (becoming more vulnerable to uncertainty). Furthermore, each robustness curve reaches the horizontal axis (zero robustness) precisely at the predicted benefit for that allocation.

Likewise, considering them as opportuneness functions, the negative slopes of the curves express a different irrevocable trade-off: as the aspired benefit, C_i, *decreases* (becoming more informative at low and unusual values), the opportuneness function for enabling that benefit *increases* (requiring greater uncertainty). Each curve reaches the horizontal axis (zero opportuneness) precisely at the predicted benefit for that allocation.

The curves in Figure 3 are evaluated with the following values for allocation among $N = 5$ categories:

$$\widetilde{S} = [1,\ 2,\ 3,\ 4,\ 5], \tag{16}$$

$$D = [5,\ 4,\ 3,\ 2,\ 1], \tag{17}$$

$$D' = [1,\ 2,\ 3,\ 4,\ 5], \tag{18}$$

$$W = \mathrm{diag}(5, 4, 3, 2, 1). \tag{19}$$

Recall that the curves in Figure 3 are both robustness and opportuneness functions, $\widehat{h}(C_e, D)$ and $\widehat{\beta}'(C_i, D)$, and that large values of robustness are preferred over small values, while small values of opportuneness are preferred over large values. Let $C_\times$ denote the value of C_e at which the curves cross one another. Thus allocation D' is robust-preferred for essential benefit exceeding $C_\times$, while D is opportune-preferred for aspired benefit exceeding $C_\times$. Thus, the antagonism between robustness and opportuneness is observed for the upper range of benefit values.

Likewise, allocation D is robust-preferred for requirements less than $C_\times$, while D' is opportune-preferred for the same range of aspirations, again demonstrating the antagonism between robustness and opportuneness for benefits less than $C_\times$.

The antagonism between robustness and opportuneness, with the associated decision dilemmas discussed earlier, arises in the present example. However, the intersection between the immunity functions causes the robust and opportune preferences to switch, depending on the range of benefits that are considered. Crossing robustness (and opportuneness) curves entails a reversal of preference between the allocations, but retains the antagonism between robustness and opportuneness.

2.6 | What type of new evidence to seek?

We now briefly touch on implications of this analysis for acquiring new evidence.

Let us suppose that we can seek evidence that may alter our estimate, $\widetilde{S}$, of the vector of returns or benefits. True evidence will improve the estimate $\widetilde{S}$. This change in $\widetilde{S}$ will (usually) either increase or decrease the value of $D^T\widetilde{S}$ and thus, correspondingly, either improve or worsen the robustness in Equation (11). However, this change in $\widetilde{S}$ will also, correspondingly, either worsen or improve the opportuneness in Equation (12). No evidence about $\widetilde{S}$ can simultaneously improve both robustness and opportuneness, or simultaneously worsen both. Evidence that reduces the impact of uncertainty by increasing $\widehat{h}(C_e)$, also reduces the ability to exploit uncertainty by increasing $\widehat{\beta}'(C_{\rm i})$. This antagonism between robustness and opportuneness presents one with a dilemma. The sequential choice of increments of evidence about $\widetilde{S}$ must address this dilemma by managing the antagonism between robustness and opportuneness.

An analogous dilemma arises if we consider evidence that alters the structure of the info-gap model of uncertainty in Equation (10), as represented by the matrix W. The axes of the ellipsoid in the info-gap model of Equation (10) are along the eigenvectors of W, denoted $v_1, \ldots, v_N$, and the length of the nth semiaxis, at horizon of uncertainty h, is $\frac{h}{\sqrt{\mu_n}}$ where μ_n is the nth eigenvalue of W. Thus, the ellipsoid at horizon of uncertainty h contracts as the eigenvalues increase: evidence that increases the eigenvalues reduces the uncertainty.

The eigenvectors of W^{-1} are the same as those of W, while the eigenvalues of W^{-1} are the inverses of those of W. Thus:

$$D^T W^{-1} D = \sum_{n=1}^{N} \frac{1}{\mu_n} D^T v_n v_n^T D. \tag{20}$$

The value of $D^T W^{-1} D$ is reduced by increasing the eigenvalues of W. From Equations (11) and (12), we see that the robustness and the opportuneness functions both increase by acquiring evidence that reduces the uncertainty by increasing the eigenvalues of W. Evidence that reduces the impact of adverse uncertainty (by enlarging $\widehat{h}$) also reduces the informativeness (by enlarging $\widehat{\beta}'$). Conversely, decreasing the robustness function also decreases the opportuneness function and increases the informativeness. Any increment of evidence cannot improve both robustness and opportuneness. This antagonism is addressed by sequentially balancing between these conflicting trends.

This antagonism between robustness and opportuneness—when identifying evidence to obtain—expresses the dilemma of uncertainty. Should one protect against adverse uncertainty by enhancing the robustness function, or should one exploit favorable uncertainty by diminishing the opportuneness function? Equivalently, should one attempt to reduce uncertainty and thereby enhance the robustness function, or should one attempt to augment uncertainty and thereby enhance the potential informativeness of new evidence? The functions of robustness and opportuneness are the same, which means that one cannot do both with the same evidence. The challenge of managing the two faces of uncertainty—adverse and favorable—entails balancing these conflicting requirements. This challenge is the dilemma one faces in managing uncertainty when selecting the next evidence to seek.

3 | FORMULATION AND PROPOSITIONS

In this section, we present a proposition and a corollary that are generalizations of the example in Section 2.

3.1 | Basic definitions

D = a decision; a choice among various options.
C = the consequence of the decision; a scalar value.
S = the system impacted by the decision.
$\widetilde{S}$ = an approximate model of system S.

The consequence is a function of the system and of the decision, so we write $C(S, D)$.

$C_{\rm e}$ is an essential or critical value of the consequence. If the outcome has adverse consequence then $C_{\rm e}$ is the greatest acceptable consequence and we require:

$$C(S, D) \leq C_{\rm e}. \tag{21}$$

If the outcome has propitious consequence, then $C_{\rm e}$ is the least acceptable consequence and we require:

$$C(S, D) \geq C_{\rm e}. \tag{22}$$

Definition of the robustness function in the subsequent propositions uses Equation (22), but these propositions can be proven based on Equation (21) with minor modification.

Various entities may be uncertain. The approximate system model $\widetilde{S}$ may err substantially and in unknown ways. The functional form of the consequence function $C(S, D)$ may err. For instance, one may assume that $C(S, D) = S^T D$ if S and D are vectors, ignoring interactions between different elements of the system. Let u denote the vector of uncertain entities (e.g., S, $C(S, D)$, etc.), where $\widetilde{u}$ is the best available estimate of u.

$\mathcal{U}(h)$ is an info-gap model for uncertainty in u. The center point is $\widetilde{u}$.

3.2 | Definitions of robustness and opportuneness

We are concerned with epistemic uncertainty: the uncertainty that confronts a decision-maker. Epistemic uncertainty is distinguished from ontological uncertainty that is an indeterminism in the real world, independent of any decision-maker.

The performance requirement in Equation (22) assumes that a large value of the consequence, $C(S, D)$, is better than a small value. The robustness of decision D is the greatest horizon of uncertainty up to which the propitious impact of the decision is guaranteed to be no less than the essential value $C_{\rm e}$:

$$\widehat{h}(C_{\rm e}, D) = \max\left\{ h : \left(\min_{u \in \mathcal{U}(h)} C(S, D) \right) \geq C_{\rm e} \right\}. \tag{23}$$

The opportuneness that is complementary to the robustness of Equation (23) is the lowest horizon of uncertainty at which it is possible that the propitious impact of the decision will be as great as the wonderfully large value $C_{\rm w}$:

$$\widehat{\beta}(C_{\rm w}, D) = \min\left\{ h : \left(\max_{u \in \mathcal{U}(h)} C(S, D) \right) \geq C_{\rm w} \right\}. \tag{24}$$

However, we will consider a different opportuneness function than Equation (24). We are interested in evidence that is informative. New insight is obtained by outcomes that are particularly severe and contrary to the anticipated outcome. Thus, the opportuneness of new evidence is the lowest horizon of uncertainty at which the outcome may be as small, severe, and informative as the value $C_{\rm i}$:

$$\hat{\beta}'(C_{\rm i}, D) = \min \left\{ h : \left(\min_{u \in \mathcal{U}(h)} C(S, D) \right) \le C_{\rm i} \right\}. \quad (25)$$

The robustness function that is the conventional complement of $\hat{\beta}'(C_{\rm i}, D)$ is the greatest horizon of uncertainty at which the consequence is guaranteed to be no larger than $C_{\rm e}$:

$$\hat{h}'(C_{\rm e}, D) = \max \left\{ h : \left(\max_{u \in \mathcal{U}(h)} C(S, D) \right) \le C_{\rm e} \right\}. \quad (26)$$

We employ $\hat{h}(C_{\rm e}, D)$ in Equation (23) and $\hat{\beta}'(C_{\rm i}, D)$ in Equation (25). We note that the inner minima in these immunity functions are the same. However, these minima are interpreted differently. In Equation (23), the inner minimum identifies the smallest—worst—outcome of the decision. The robustness, $\hat{h}(C_{\rm e}, D)$, is the greatest horizon of uncertainty up to which worse outcomes are guaranteed not to occur. In Equation (25), the inner minimum identifies the smallest—most informative—outcome of the decision. The opportuneness, $\hat{\beta}'(C_{\rm i}, D)$, is the lowest horizon of uncertainty at which such informatively surprising outcomes can occur.

The robustness and opportuneness functions are both immunity functions. $\hat{h}(C_{\rm e}, D)$ is the immunity against uncertainty that would cause unacceptably poor outcome. $\hat{\beta}'(C_{\rm i}, D)$ is the immunity against uncertainty that could enable a wonderfully useful and informative outcome. A large value of $\hat{h}(C_{\rm e}, D)$ is desirable, while a small value of $\hat{\beta}'(C_{\rm i}, D)$ is desirable.

3.3 | Proposition and corollary

Proposition 1. *Robustness and opportuneness are antagonistic: A change in the decision that improves one at a specific performance value, worsens the other at this value.*

Given:

- *An info-gap model, $\mathcal{U}(h)$, obeying the axioms of nesting and contraction.*
- *A robustness function, $\hat{h}(C_{\rm e}, D)$, defined in Equation (23).*
- *An opportuneness function, $\hat{\beta}'(C_{\rm i}, D)$, defined in Equation (25).*
- *Two decisions, D_1 and D_2.*
- *A performance value, $C_\star$, which may be adopted for either $C_{\rm e}$ or $C_{\rm i}$.*

Then, D_2 is more robust than D_1 at $C_\star$ if and only if D_2 is less opportune than D_1 at $C_\star$. Specifically:

$$\hat{h}(C_{\rm e}, D_1) \le \hat{h}(C_{\rm e}, D_2) \quad \text{for} \quad C_{\rm e} = C_\star \quad (27)$$

if and only if:

$$\hat{\beta}'(C_{\rm i}, D_1) \le \hat{\beta}'(C_{\rm i}, D_2) \quad \text{for} \quad C_{\rm i} = C_\star. \quad (28)$$

Proof of Proposition 1. Define $m(h, D)$ as:

$$m(h, D) = \min_{u \in \mathcal{U}(h)} C(S, D) \quad \text{for all} \quad h \ge 0. \quad (29)$$

$m(h, D)$ is the inner minimum in the definitions of the immunity functions in Equations (23) and (25).

$m(h, D_k)$ is the inverse function of $\hat{h}(C_{\rm e}, D_k)$ for $k = 1, 2$. That is, a plot of h versus $m(h, D_k)$ is identical to a plot of $\hat{h}(C_{\rm e}, D_k)$ versus $C_{\rm e}$.

Likewise, $m(h, D_k)$ is the inverse function of $\hat{\beta}'(C_{\rm i}, D_k)$ for $k = 1, 2$. That is, a plot of h versus $m(h, D_k)$ is identical to a plot of $\hat{\beta}'(C_{\rm i}, D_k)$ versus $C_{\rm i}$.

In other words, the curve $\hat{h}(C_{\rm e}, D_k)$ versus $C_{\rm e}$ is identical to the curve $\hat{\beta}'(C_{\rm i}, D_k)$ versus $C_{\rm i}$ because their inverse functions are identical.

The equivalence of Equations (27) and (28) results immediately. □

Proposition 1 establishes the antagonism between robustness and opportuneness at any specific performance value for $C_{\rm e}$ and $C_{\rm i}$. The proposition does not presume that D_2 is robust dominant over D_1, namely, it does not assume that $\hat{h}(C_{\rm e}, D_2)$ exceeds $\hat{h}(C_{\rm e}, D_1)$ throughout the domain of $C_{\rm e}$. These robustness curves may in fact cross one another. The proposition asserts that at any performance value where one decision is more robust than the other, that decision is also less opportune than the other at the same performance value.

The following corollary derives immediately from Proposition 1.

Corollary 1. *Robustness and opportuneness are antagonistic: Robust dominance and opportune dominance are opposites.*

Given:

- *An info-gap model, $\mathcal{U}(h)$, obeying the axioms of nesting and contraction.*
- *A robustness function, $\hat{h}(C_{\rm e}, D)$, defined in Equation (23).*
- *An opportuneness function, $\hat{\beta}'(C_{\rm i}, D)$, defined in Equation (25).*
- *Two decisions, D_1 and D_2.*

Then, D_2 is robust dominant over D_1 if and only if D_1 is opportune dominant over D_2. Specifically:

$$\hat{h}(C_{\rm e}, D_1) \le \hat{h}(C_{\rm e}, D_2) \quad \text{for all} \quad C_{\rm e} \quad (30)$$

if and only if:

$$\widehat{\beta}'(C_\mathrm{i}, D_1) \le \widehat{\beta}'(C_\mathrm{i}, D_2) \quad \text{for all} \quad C_\mathrm{i}. \tag{31}$$

Proof of Corollary 1. The proof derives immediately by applying Proposition 1 throughout the domain of definition of the performance values C_e and C_i. □

4 | DISCUSSION AND CONCLUSION

Decision making based on evidence and understanding is standard in many disciplines. True evidence can reduce uncertainty, enhance understanding, and lead to reliable and responsible decisions. However, true evidence can augment one's uncertainty by uncovering mistaken or deficient prior knowledge. Uncertainty-augmenting evidence is important because it also can lead to improved understanding, better decisions, and exploitation of previously unrecognized opportunities.

Evidence can either reduce or enhance the uncertainty that confronts a decision-maker. The dilemma facing the decision-maker is that both reduction and augmentation of uncertainty have desirable attributes because uncertainty can be pernicious or propitious: threatening failure or enabling windfall, respectively.

Different methodologies are appropriate for decision making when facing these two modes of uncertainty. Pernicious uncertainty can be managed with the info-gap robustness function that satisfices the outcome and maximizes the immunity to surprise. Propitious uncertainty can be managed with the info-gap opportuneness function that facilitates better-than-anticipated windfall outcome.

Decision-makers seek to manage both pernicious and propitious uncertainty, but this creates a dilemma. Robustness and opportuneness impose conflicting demands: a decision that enhances one, worsens the other. No decision can improve robustness against surprise as well as opportuneness from surprise. No decision can both protect against adverse uncertainty and exploit favorable uncertainty. This is demonstrated in a generic proposition and its corollary, and illustrated in an example of allocation of limited resources for achieving collective benefit.

Recognizing and quantifying the conflict between robustness and opportuneness enables the decision-maker to balance between two goals: robustly satisficing critical requirements, or opportunely enabling potential windfalls. The balance is not unique, and the decision-maker selects the balance according to the goals.

Feduzi (2010) discusses "the 'stopping problem' of finding a rational principle to decide where to stop the process of acquiring information in forming a probability judgment before making a decision" (p. 338). Feduzi argues that the stopping problem "sheds light on… the decision maker's subjective assessment of relevant ignorance in the process of rational decision making" (p. 350). Future research can explore how the balancing between robustness and opportuneness can assist in both characterizing and supporting decision-makers' assessments in addressing the stopping problem.

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