87. **Quantiles with asymmetric uncertainty,** (p.314) x is a non-negative random variable with probability density function (pdf) $p(x)$. The system we are designing will fail if x is too large. We want to know the largest value of x for which the probability of not exceeding this value is $1-\alpha$. This value is called the $(1-\alpha)$ quantile of x, denoted q_α, and defined in the relation:

$$1-\alpha = \int_0^{q_\alpha} p(x)\,\mathrm{d}x \tag{404}$$

(a) Derive an explicit algebraic expression for the $(1-\alpha)$ quantile of x using the exponential distribution:

$$\widetilde{p}(x) = \widetilde{\lambda}\mathrm{e}^{-\widetilde{\lambda}x} \tag{405}$$

(b) Now suppose that the true pdf of x, denoted $p(x)$, is exponential but the coefficient of the distribution, λ, is uncertain. The best available estimate is $\widetilde{\lambda}$ (which is positive) but we suspect that this is an under estimate. We represent the uncertainty in the pdf of x with this info-gap model:

$$\mathcal{U}(h) = \left\{ p(x) = \lambda\mathrm{e}^{-\lambda x} :\ 0 \le \frac{\lambda-\widetilde{\lambda}}{s} \le h \right\}, \quad h \ge 0 \tag{406}$$

where s is a known positive constant. We will estimate the $(1-\alpha)$ quantile using $\widetilde{p}(x)$ in eq.(405), but this will be an over estimate (explain why):

$$0 \le q_\alpha(p) \le q_\alpha(\widetilde{p}) \tag{407}$$

We require that this over estimate not err by more than ε:

$$q_\alpha(\widetilde{p}) - q_\alpha(p) \le \varepsilon \tag{408}$$

Derive an explicit algebraic expression for the robustness if we estimate the quantile as $q_\alpha(\widetilde{p})$.

(c) We continue with the info-gap model of eq.(406) but we estimate the quantile with an exponential distribution whose coefficient, λ_e, is greater than $\widetilde{\lambda}$. For convenience we will denote quantiles according to the exponential coefficient, so our estimate of the quantile is $q_\alpha(\lambda_\mathrm{e})$ and we require that the absolute error of this estimate not exceed ε:

$$|q_\alpha(\lambda_\mathrm{e}) - q_\alpha(\lambda)| \le \varepsilon \tag{409}$$

Derive an algebraic expression for the inverse of the robustness function. Explore the crossing of these robustness curves with the robustness curve of part 87b.